

## RANDOM CONFORMAL WELDINGS

KARI ASTALA<sup>1</sup>, PETER JONES, ANTTI KUPIAINEN<sup>1</sup>, AND EERO SAKSMAN<sup>1</sup>

ABSTRACT. We construct a conformally invariant random family of closed curves in the plane by welding of random homeomorphisms of the unit circle. The homeomorphism is constructed using the exponential of  $\beta X$  where  $X$  is the restriction of the two dimensional free field on the circle and the parameter  $\beta$  is in the "high temperature" regime  $\beta < \sqrt{2}$ . The welding problem is solved by studying a non-uniformly elliptic Beltrami equation with a random complex dilatation. For the existence a method of Lehto is used. This requires sharp probabilistic estimates to control conformal moduli of annuli and they are proven by decomposing the free field as a sum of independent fixed scale fields and controlling the correlations of the complex dilation restricted to dyadic cells of various scales. For uniqueness we invoke a result by Jones and Smirnov on conformal removability of Hölder curves. We conjecture that our curves are locally related to  $\text{SLE}(\kappa)$  for  $\kappa < 4$ .

## 1. INTRODUCTION

There has been great interest in conformally invariant random curves and fractals in the plane ever since it was realized that such geometric objects appear naturally in statistical mechanics models at the critical temperature [8]. A major breakthrough in the field occurred when O. Schramm [28] introduced the Schramm-Loewner Evolution (SLE), a stochastic process whose sample paths are conjectured (and in several cases proved) to be the curves occurring in the physical models. We refer to [29] and [30] for a general overview and some recent work on SLE. SLE curves come in two varieties: the radial one where the curve joins a boundary point (say of the disc) to an interior point and the chordal case where two boundary points are joined.

SLE describes a curve growing in time: the original curve of interest (say a cluster boundary in a spin system) is obtained as time tends to infinity. In this paper we give a different construction of random curves which is stationary i.e. the probability measure on curves is directly defined without introducing an auxiliary time. We carry out this construction for closed curves, a case that is not naturally covered by SLE.

Our construction is based on the idea of conformal welding. Consider a Jordan curve  $\gamma$  bounding a simply connected region  $\Omega$  in the plane. By the Riemann mapping theorem there are conformal maps  $f_{\pm}$  mapping the unit disc  $\mathbb{D}$  and its complement to  $\Omega$  and its complement. The map  $f_+^{-1} \circ f_-$  extends continuously to the boundary  $\mathbb{T} = \partial\mathbb{D}$  of the disc and defines a homeomorphism of the circle. Conformal Welding is the inverse operation where, given a suitable homeomorphism of the circle, one constructs a Jordan curve on the plane (see Section 2). In fact, in our case the curve is determined up to a Möbius transformation of the plane. Thus random curves (modulo Möbius transformations) can be obtained from random homeomorphisms via welding.

---

<sup>1</sup>Supported by the Academy of Finland

*Key words and phrases.* Random welding, quasi-conformal maps, SLE.

In this paper we introduce a random scale invariant set of homeomorphisms  $h_\omega : \mathbb{T} \rightarrow \mathbb{T}$  and construct the welding curves. The model considered here has been proposed by the second author. The construction depends on a real parameter  $\beta$  ("inverse temperature") and the maps are a.s. in  $\omega$  Hölder continuous for  $\beta < \beta_c$ . For this range of  $\beta$  the welding map will be a.s. well-defined. For  $\beta > \beta_c$  we expect the map  $h_\omega$  not to be continuous and no welding to exist. We conjecture that the resulting curves should locally "look like" SLE( $\kappa$ ) for  $\kappa < 4$  but we don't have good arguments for this. The case  $\beta = \beta_c$ , presumably corresponding to SLE(4), is not covered by our analysis.

Since we are interested in random curves that are self similar it is natural to take  $h$  with such properties. Our choice for  $h$  is constructed by starting with the Gaussian random field  $X$  on the circle (see Section 3 for precise definitions) with covariance

$$(1) \quad \mathbb{E} X(z)X(z') = -\log |z - z'|$$

where  $z, z' \in \mathbb{C}$  with modulus one.  $X$  is just the restriction of the 2d massless free field (Gaussian Free Field) on the circle. The exponential of  $\beta X$  gives rise to a random measure  $\tau$  on the unit circle  $\mathbb{T}$ , formally given by

$$(2) \quad "d\tau = e^{\beta X(z)} dz".$$

The proper definition involves a limiting process  $\tau(dz) = \lim_{\varepsilon \rightarrow 0} e^{\beta X_\varepsilon(z)} / \mathbb{E} e^{\beta X_\varepsilon(z)} dz$ , where  $X_\varepsilon$  stands for a suitable regularization of  $X$ , see Section 3.3 below.

Identifying the circle as  $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1)$  our random homeomorphism  $h : [0, 1) \rightarrow [0, 1)$  is defined as

$$(3) \quad h(\theta) = \tau([0, \theta)) / \tau([0, 1)) \quad \text{for } \theta \in [0, 1).$$

The main result of this paper can then be summarized as follows:

*For  $\beta^2 < 2$  and almost surely in  $\omega$ , the formula (3) defines a Hölder continuous circle homeomorphism, such that the welding problem has a solution  $\gamma$ , where  $\gamma$  is a Jordan curve bounding a domain  $\Omega = f_+(\mathbb{D})$  with a Hölder continuous Riemann mapping  $f_+$ . For a given  $\omega$ , the solution is unique up to a Möbius map of the plane.*

We refer to Section 5 (Theorem 5.2) for the exact statement of the main result.

Apart from connection to SLE the weldings constructed in this paper should be of interest to complex analysts as they form a natural family that degenerates as  $\beta \uparrow \sqrt{2}$ . It would be of great interest to understand the critical case  $\beta = \sqrt{2}$  as well as the low temperature "spin glass phase"  $\beta > \sqrt{2}$ . It would also be of interest to understand the connection of our weldings to those arising from stochastic flows [1]. In [1] Hölder continuous homeomorphisms are considered, but the boundary behaviour of the welding and hence its existence and uniqueness are left open.

In writing the paper we have tried be generous in providing details on both the function theoretic and the stochastics notions and tools needed, in order to serve readers with varied backgrounds. The structure of the paper is as follows. Section 2 contains background material on conformal welding and the geometric analysis tools we need later on. To be more specific, Section 2 recalls the notion of conformal welding and explains how the welding problem is reduced to the study of the Beltrami equation. Also we recall a useful method due to Lehto [21] to prove the existence of a solution for a class of non-uniformly elliptic Beltrami equations, and a theorem by Jones and Smirnov [15] that will be used to verify the uniqueness of our welding. Finally we

recall the Beurling-Ahlfors extension of homeomorphisms of the circle to the inside the disc. For our purposes we need to estimate carefully the dependence of the dilatation of the extension in a Whitney cube by just using small amount of information of the homeomorphism on a 'shadow' of the cube.

In Section 3 we introduce the one-dimensional trace of the Gaussian free field and recall some known properties of its exponential that we will use to define and study the random circle homeomorphism. Section 4 is the technical core of the paper as it contains the main probabilistic estimates we need to control the random dilatation of the extension map. Finally, in Section 5 things are put together and the a.s. existence and uniqueness of the welding map is proven.

Let us finish by a remark on notation. We denote by  $c$  and  $C$  generic constants which may vary between estimates. When the constants depend on parameters such as  $\beta$  we denote this by  $C(\beta)$ .

**Acknowledgements.** We thank M. Bauer, D. Bernard, Steffen Rohde and Stanislav Smirnov for useful discussions. This work is partially funded by the Academy of Finland, European Research Council and National Science Foundation.

## 2. CONFORMAL WELDING

In the present section we recall for the general readers benefit basic notions and results from geometric analysis that are needed in our work. In particular, we recall the notion of conformal welding, Lehto's method for solving the Beltrami-equations, the uniqueness result for weldings due to Jones and Smirnov, and the last subsection contains estimates for the Beurling-Ahlfors extension tailored for our needs.

**2.1. Welding and Beltrami equation.** One of the main methods for constructing conformally invariant families of (Jordan) curves comes from the theory of *conformal welding*. Put briefly, in this method we glue the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  and the exterior disk  $\mathbb{D}_\infty = \{z \in \widehat{\mathbb{C}} : |z| > 1\}$  along a homeomorphism  $\phi : \mathbb{T} \rightarrow \mathbb{T}$ , by the identification

$$x \sim y, \quad \text{when } x \in \mathbb{T} = \partial\mathbb{D}, \quad y = \phi(x) \in \mathbb{T} = \partial\mathbb{D}_\infty.$$

The problem of welding is to give a natural complex structure to this topological sphere. Uniformizing the complex structure then gives us the curve, the image of the unit circle.

More concretely, given a Jordan curve  $\Gamma \subset \widehat{\mathbb{C}}$ , let

$$f_+ : \mathbb{D} \rightarrow \Omega_+ \quad \text{and} \quad f_- : \mathbb{D}_\infty \rightarrow \Omega_-$$

be a choice of Riemann mappings onto the components of the complement  $\widehat{\mathbb{C}} \setminus \Gamma = \Omega_+ \cup \Omega_-$ . By Caratheodory's theorem  $f_-$  and  $f_+$  both extend continuously to  $\partial\mathbb{D} = \partial\mathbb{D}_\infty$ , and thus

$$(4) \quad \phi = (f_+)^{-1} \circ f_-$$

is a homeomorphism of  $\mathbb{T}$ . In the welding problem we are asked to invert this process; given a homeomorphism  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  we are to find a Jordan curve  $\Gamma$  and conformal mappings  $f_\pm$  onto the complementary domains  $\Omega_\pm$  so that (4) holds.

It is clear that the welding problem, when solvable, has natural conformal invariance attached to it; any image of the curve  $\Gamma$  under a Möbius transformation of  $\widehat{\mathbb{C}}$  is equally

a welding curve. Similarly, if  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  admits a welding, then so do all its compositions with Möbius transformations of the disk. Note, however, that not all circle homeomorphisms admit a welding, for examples see [24] and [32].

The most powerful tool in solving the welding problem is given by the Beltrami differential equation, defined in a domain  $\Omega$  by

$$(5) \quad \frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z}, \quad \text{for almost every } z \in \Omega,$$

where we look for homeomorphic solutions  $f \in W_{loc}^{1,1}(\Omega)$ . Here (5) is an elliptic system whenever  $|\mu(z)| < 1$  almost everywhere, and uniformly elliptic if there is a constant  $0 \leq k < 1$  such that  $\|\mu\|_\infty \leq k$ .

In the uniformly elliptic case, homeomorphic solutions to (5) exist for every coefficient with  $\|\mu\|_\infty < 1$ , and they are unique up to post-composing with a conformal mapping [4, p.179]. In fact, it is this uniqueness property that gives us a way to produce the welding. To see this suppose first that

$$(6) \quad \phi = f|_{\mathbb{T}},$$

where  $f \in W_{loc}^{1,2}(\mathbb{D}; \mathbb{D}) \cap C(\overline{\mathbb{D}})$  is a homeomorphic solution to (5) in the disc  $\mathbb{D}$ . Find then a homeomorphic solution to the auxiliary equation

$$(7) \quad \frac{\partial F}{\partial \bar{z}} = \chi_{\mathbb{D}}(z) \mu(z) \frac{\partial F}{\partial z}, \quad \text{for a.e. } z \in \mathbb{C}.$$

Now  $\Gamma = F(\mathbb{T})$  is a Jordan curve. Moreover, as  $\partial_{\bar{z}} F = 0$  for  $|z| > 1$ , we can set  $f_- := F|_{\mathbb{D}_\infty}$  and  $\Omega_- := F(\mathbb{D}_\infty)$  to define a conformal mapping

$$f_- : \mathbb{D}_\infty \rightarrow \Omega_-$$

On the other hand, since both  $f$  and  $F$  solve the equation (5) in the unit disk  $\mathbb{D}$ , by uniqueness of the solutions we have

$$(8) \quad F(z) = f_+ \circ f(z), \quad z \in \mathbb{D},$$

for some conformal mapping  $f_+ : \mathbb{D} = f(\mathbb{D}) \rightarrow \Omega_+ := F(\mathbb{D})$ . Finally, on the unit circle,

$$(9) \quad \phi(z) = f|_{\mathbb{T}}(z) = (f_+)^{-1} \circ f_-(z), \quad z \in \mathbb{T}.$$

Thus we have found a solution to the welding problem, under the assumption (6). That the welding curve  $\Gamma$  is unique up to a Möbius transformation of  $\mathbb{C}$  follows from [4, Theorem 5.10.1], see also Corollary 2.5 below.

To complete this circle of ideas we need to identify the homeomorphisms  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  that admit the representation (5), (6) with uniformly elliptic  $\mu$  in (5). It turns out [4, Lemma 3.11.3 and Theorem 5.8.1] that such  $\phi$ 's are precisely the quasymmetric mappings of  $\mathbb{T}$ , mappings that satisfy

$$(10) \quad K(\phi) := \sup_{s,t \in \mathbb{R}} \frac{|\phi(e^{2\pi i(s+t)}) - \phi(e^{2\pi is})|}{|\phi(e^{2\pi i(s-t)}) - \phi(e^{2\pi is})|} < \infty.$$

**2.2. Existence in the degenerate case: the Lehto condition.** The previous subsection describes an obvious model for constructing random Jordan curves, by first finding random homeomorphisms of the circle and then solving for each of them the associated welding problem. In the present work, however, we are faced with the obstruction that circle homeomorphisms with derivative the exponentiated Gaussian free field almost surely do not satisfy the quasimetric assumption (10). Thus we are forced outside the uniformly elliptic PDE's and need to study (5) with degenerate coefficients with only  $|\mu(z)| < 1$  almost everywhere. We are even outside the much studied class of maps of exponentially integrable distortion, see [4, 20.4.] In such generality, however, the homeomorphic solutions to (5) may fail to exist, or the crucial uniqueness properties of (5) may similarly fail.

In his important work [21] Lehto gave a very general condition in the degenerate setting, for the existence of homeomorphic solutions to (5). To recall his result, assume we are given the complex dilatation  $\mu = \mu(z)$ , and write then

$$K(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}, \quad z \in \Omega,$$

for the associated distortion function. Note that  $K(z)$  is bounded precisely when the equation (5) is uniformly elliptic, i.e.  $\|\mu\|_\infty < 1$ . Thus the question is how strongly can  $K(z)$  grow for the basic properties of (5) still to remain true. In order to state Lehto's condition we fix some notation. For given  $z \in \mathbb{C}$  and radii  $0 \leq r < R < \infty$  let us denote the corresponding annulus by

$$A(z, r, R) := \{w \in \mathbb{C} : r < |w - z| < R\}.$$

In the Lehto approach one needs to control the conformal moduli of image annuli in a suitable way. This is done by introducing for any annulus  $A(w, r, R)$  and for the given distortion function  $K$  the following quantity, which we call *the Lehto integral*:

$$(11) \quad L(z, r, R) := L_K(z, r, R) := \int_r^R \frac{1}{\int_0^{2\pi} K(z + \rho e^{i\theta}) d\theta} \frac{d\rho}{\rho}$$

For the following formulation of Lehto's theorem see [4, p. 584].

**Theorem 2.1.** Suppose  $\mu$  is measurable and compactly supported with  $|\mu(z)| < 1$  for almost every  $z \in \mathbb{C}$ . Assume that the distortion function  $K(z) = (1 + |\mu(z)|)/(1 - |\mu(z)|)$  is locally integrable,

$$(12) \quad K \in L_{loc}^1(\mathbb{C}),$$

and that for some  $R_0 > 0$  the Lehto integral satisfies

$$(13) \quad L_K(z, 0, R_0) = \infty, \quad \text{for all } z \in \mathbb{C}.$$

Then the Beltrami equation

$$(14) \quad \frac{\partial f}{\partial \bar{z}}(z) = \mu(z) \frac{\partial f}{\partial z}(z) \quad \text{for almost every } z \in \mathbb{C},$$

admits a homeomorphic  $W_{loc}^{1,1}$ -solution  $f : \mathbb{C} \rightarrow \mathbb{C}$ .

**Corollary 2.2.** Suppose  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  extends to a homeomorphism  $f : \mathbb{C} \rightarrow \mathbb{C}$  satisfying (12) - (14) together with the condition

$$(15) \quad K(z) \in L_{loc}^\infty(\mathbb{D}).$$

Then  $\phi$  admits a welding: there are a Jordan curve  $\Gamma \subset \widehat{\mathbb{C}}$  and conformal mappings  $f_\pm$  onto the complementary domains of  $\Gamma$  such that

$$\phi(z) = (f_+)^{-1} \circ f_-(z), \quad z \in \mathbb{T}.$$

**Proof.** Given the extension  $f : \mathbb{C} \rightarrow \mathbb{C}$  let us again look at the auxiliary equation

$$(16) \quad \frac{\partial F}{\partial \bar{z}} = \chi_{\mathbb{D}}(z) \mu(z) \frac{\partial F}{\partial z}, \quad \text{for a.e. } z \in \mathbb{C}.$$

Since Lehto's condition holds as well for the new distortion function

$$K(z) = \frac{1 + |\chi_{\mathbb{D}}(z) \mu(z)|}{1 - |\chi_{\mathbb{D}}(z) \mu(z)|},$$

we see from Theorem 2.1 that the auxiliary equation (16) admits a homeomorphic solution  $F : \mathbb{C} \rightarrow \mathbb{C}$ . Arguing as in (6) - (9) it will be then sufficient to show that

$$F(z) = f_+ \circ f(z), \quad z \in \mathbb{D},$$

where  $f_+$  is conformal in  $\mathbb{D}$ . But this is a local question; every point  $z \in \mathbb{D}$  has a neighborhood where  $K(z)$  is uniformly bounded, by (15). In such a neighborhood the usual uniqueness results to solutions of (5) apply, see [4, p.179]. Thus  $f_+$  is holomorphic, and as a homeomorphism it is conformal. This proves the claim.  $\square$

Consequently, in the study of random circle homeomorphisms  $\phi = \phi_\omega$  a key step for the conformal welding of  $\phi_\omega$  will be to show that almost surely each such mapping admits a homeomorphic extension to  $\mathbb{C}$ , where the distortion function satisfies a condition such as (13). In our setting where derivative of  $\phi$  is given by the exponentiated trace of a Gaussian free field, the extension procedure is described in Section 2.4 and the appropriate estimates it requires are proven in Section 4.

Actually, in Section 5 when proving our main theorem we need to present a variant of Lehto's argument where it will be enough to estimate the Lehto integral only at a suitable countable set of points  $z \in \mathbb{T}$ . We also utilize there the fact that the extension of our random circle homeomorphism  $\phi$  satisfies (15). In verifying the Hölder continuity of the ensuing map we shall apply a useful estimate (Lemma 2.3 below) that estimates the geometric distortion of an annulus under a quasiconformal map.

Given a bounded (topological) annulus  $A \subset \mathbb{C}$ , with  $E$  the bounded component of  $\mathbb{C} \setminus A$ , we denote by  $D_O(A) := \text{diam}(A)$  the outer diameter, and by  $D_I(A) := \text{diam}(E)$  the inner diameter of  $A$ .

**Lemma 2.3.** *Let  $f$  be a quasiconformal mapping on the annulus  $A(w, r, R)$ , with the distortion function  $K_f$ . It then holds that*

$$\frac{D_O(f(A(w, r, R)))}{D_I(f(A(w, r, R)))} \geq \frac{1}{16} \exp(2\pi^2 L_{K_f}(w, r, R)).$$

**Proof.** Recall first that for a rigid annulus  $A = A(w, r, R)$  the modulus

$$\text{mod}(A) = 2\pi \log \frac{R}{r}$$

while for any topological annulus  $A$ , we define its conformal modulus by  $\text{mod}(A) = \text{mod}(g(A))$  where  $g$  is a conformal map of  $A$  onto a rigid annulus. Then we have [4, Cor. 20.9.2] the following basic estimate for the modulus of the image annulus in terms of the Lehto integrals:

$$(17) \quad \text{mod}(f(A(w, r, R))) \geq 2\pi L_{K_f}(w, r, R).$$

On the other hand, by combining [33, 7.38 and 7.39] and [3, 5.68(16)] we obtain for any bounded topological annulus  $A \subset \mathbb{C}$

$$\frac{1}{16} \exp(\pi \text{mod}(A)) \leq \frac{D_O(A)}{D_I(A)}.$$

Put together, the desired estimate follows.  $\square$

**2.3. Uniqueness of the welding.** An important issue of the welding is its uniqueness, that the curve  $\Gamma$  is unique up to composing with a Möbius transformation of  $\widehat{\mathbb{C}}$ . As the above argument indicates, this is essentially equivalent to the uniqueness of solutions to the appropriate Beltrami equations, up to a Möbius transformation. However, in general the assumptions of Theorem 2.1 alone are much too weak to imply this.

It fact, in our case the uniqueness of solutions to the Beltrami equation (16) is equivalent to the conformal removability of the curve  $F(\mathbb{T})$ . Recall that a compact set  $B \subset \widehat{\mathbb{C}}$  is *conformally removable* if every homeomorphism of  $\widehat{\mathbb{C}}$  which is conformal off  $B$  is conformal in the whole sphere, hence a Möbius transformation.

It follows easily that e.g. images of circles under quasiconformal mappings, i.e. homeomorphisms satisfying (19) with  $\|\mu\|_\infty < 1$ , are conformally removable, while Jordan curves of positive area are never conformally removable.

For general curves the removability is a deep problem; no characterizations of conformally removable Jordan curves is known to this date. What saves us in the present work is that we have available the remarkable result of Jones and Smirnov in [15]. We will not need their result in its full generality, as the following special case will sufficient for our purposes.

**Theorem 2.4.** (Jones, Smirnov [15]) Let  $\Omega \subset \widehat{\mathbb{C}}$  be a simply connected domain such that the Riemann mapping  $\psi : \mathbb{D} \rightarrow \Omega$  is  $\alpha$ -Hölder continuous for some  $\alpha > 0$ .

Then the boundary  $\partial\Omega$  is conformally removable.

Adapting this result to our setting we obtain

**Corollary 2.5.** Suppose  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  is a homeomorphism that admits a welding

$$\phi(z) = (f_+)^{-1} \circ f_-(z), \quad z \in \mathbb{T},$$

where  $f_\pm$  are conformal mappings of  $\mathbb{D}$  and  $\mathbb{D}_\infty$ , respectively, onto complementary Jordan domains  $\Omega_\pm$ .

Assume that  $f_-$  (or  $f_+$ ) is  $\alpha$ -Hölder continuous on the boundary  $\partial\mathbb{D}_\infty = \mathbb{T}$ . Then the welding is unique: any other welding pair  $(g_+, g_-)$  of  $\phi$  is of the form

$$g_\pm = \Phi \circ f_\pm, \quad \Phi : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}} \text{ Möbius.}$$

**Proof.** Suppose we have Riemann mappings  $g_\pm$  onto complementary Jordan domains such that

$$(g_+)^{-1} \circ g_-(z) = \phi(z) = (f_+)^{-1} \circ f_-(z), \quad z \in \mathbb{T}.$$

Then the formula

$$\Psi(z) = \begin{cases} g_+ \circ (f_+)^{-1}(z) & \text{if } z \in f_+(\mathbb{D}) \\ g_- \circ (f_-)^{-1}(z) & \text{if } z \in f_-(\mathbb{D}_\infty) \end{cases}$$

defines a homeomorphism of  $\widehat{\mathbb{C}}$  that is conformal outside  $\Gamma = f_\pm(\mathbb{T})$ . From the Jones-Smirnov theorem we see that  $\Psi$  extends conformally to the entire sphere; thus it is a Möbius transformation.  $\square$

As we shall see in Theorem 5.1, for circle homeomorphisms  $\phi$  with derivative the exponentiated Gaussian free field, the solutions  $F$  to the auxiliary equation (16) will be Hölder continuous almost surely. Then  $f_- = F|_{\mathbb{D}_\infty}$  is a Riemann mapping onto a complementary component of the welding curve of  $\phi = \phi_\omega$ . It follows that almost surely the  $\phi = \phi_\omega$  admits a welding curve  $\Gamma = \Gamma_\omega$  which is unique, up to composing with a Möbius transformation.

**2.4. Extension of the homeomorphism.** In this section we discuss in detail suitable methods of extending homeomorphisms  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  to the unit disk; by reflecting across  $\mathbb{T}$  the map then extends to  $\mathbb{C}$ . Extensions of homeomorphisms  $h : \mathbb{R} \rightarrow \mathbb{R}$  of the real line are convenient to describe, and it is not difficult to find constructions that sufficiently well respect the conformally invariant features of  $h$ . Given a homeomorphism  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  on the circle, we hence represent it in the form

$$(18) \quad \phi(e^{2\pi i x}) = e^{2\pi i h(x)}$$

where  $h : \mathbb{R} \rightarrow \mathbb{R}$  is a homeomorphism of the line with  $h(x+1) = h(x) + 1$ . We may assume that  $\phi(1) = 1$ , with  $h(0) = 0$ .

We will now extend the 1-periodic mapping  $h$  to the upper (or lower) half plane so that it becomes the identity map at large height. Then a conjugation to a mapping of the disk is easily done. For the extension we use the classical Beurling-Ahlfors extension [6] modified suitably far away from the real axis.

Thus, given a homeomorphism  $h : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$(19) \quad h(x+1) = h(x) + 1, \quad x \in \mathbb{R}, \quad \text{with } h(0) = 0,$$

we define our extension  $F$  as follows. For  $0 < y < 1$  let

$$(20) \quad \begin{aligned} F(x+iy) &= \frac{1}{2} \int_0^1 (h(x+ty) + h(x-ty)) dt \\ &\quad + i \int_0^1 (h(x+ty) - h(x-ty)) dt. \end{aligned}$$

Then  $F = h$  on the real axis, and  $F$  is a continuously differentiable homeomorphism.



Moreover, by (19) it follows that for  $y = 1$ ,

$$F(x + i) = x + i + c_0,$$

where  $c_0 = \int_0^1 h(t)dt - 1/2 \in [-1/2, 1/2]$ . Thus for  $1 \leq y \leq 2$  we set

$$(21) \quad F(z) = z + (2 - y)c_0,$$

and finally have an extension of  $h$  with the extra properties

$$(22) \quad F(z) \equiv z \quad \text{when } y = \Im m(z) \geq 2,$$

$$(23) \quad F(z + k) = F(z) + k, \quad k \in \mathbb{Z}.$$

The original circle mapping admits a natural extension to the disk,

$$(24) \quad \Psi(z) = \exp(2\pi i F(\log z / 2\pi i)), \quad z \in \mathbb{D}.$$

From (18), (23) we see that this is a well defined homeomorphism of the disk with  $\Psi|_{\mathbb{T}} = \phi$  and  $\Psi(z) \equiv z$  for  $|z| \leq e^{-4\pi}$ . The distortion properties are not altered under this locally conformal change of variables,

$$(25) \quad K(z, \Psi) = K(w, F), \quad z = e^{2\pi i w}, \quad w \in \mathbb{H},$$

so we will reduce all distortion estimates for  $\Psi$  to the corresponding ones for  $F$ . Since  $F$  is conformal for  $y > 2$  it suffices to restrict the analysis to the strip

$$(26) \quad S = \mathbb{R} \times [0, 2].$$

To estimate  $K(w, F)$  we introduce some notation. Let

$$\mathcal{D}_n = \{ [k2^{-n}, (k+1)2^{-n}] : k \in \mathbb{Z} \}$$

be the set of all dyadic intervals of length  $2^{-n}$  and write

$$\mathcal{D} = \{ \mathcal{D}_n : n \geq 0 \}.$$

Consider the measure

$$\tau([a, b]) = h(b) - h(a).$$

For a pair of intervals  $\mathbf{J} = \{J_1, J_2\}$  let us introduce the following quantity

$$(27) \quad \delta_\tau(\mathbf{J}) = \tau(J_1)/\tau(J_2) + \tau(J_2)/\tau(J_1).$$

If  $J_i$  are the two halves of an interval  $I$ , then  $\delta_\tau(\mathbf{J})$  measures the local doubling properties of the measure  $\tau$ . In such a case we define  $\delta_\tau(I) = \delta_\tau(\mathbf{J})$ . In particular, (10) holds for the circle homeomorphism  $\phi(e^{2\pi i x}) = e^{2\pi i h(x)}$  if and only if the quantities  $\delta_\tau(I)$  are uniformly bounded, for all (not necessarily dyadic) intervals  $I$ .

The local distortion of the extension  $F$  will be controlled by sums of the expressions  $\delta_\tau(\mathbf{J})$  in the appropriate scale. For this, let us pave the strip  $S$  by Whitney cubes  $\{C_I\}_{I \in \mathcal{D}}$  defined by

$$C_I = \{(x, y) : x \in I, 2^{-n-1} \leq y \leq 2^{-n}\}$$

for  $I \in \mathcal{D}_n$ ,  $n > 0$  and  $C_I = I \times [\frac{1}{2}, 2]$  for  $I \in \mathcal{D}_0$ . Given an  $I \in \mathcal{D}_n$  let  $j(I)$  be the union of  $I$  and its neighbours in  $\mathcal{D}_n$  and

$$(28) \quad \mathcal{J}(I) := \{\mathbf{J} = (J_1, J_2) : J_i \in \mathcal{D}_{n+5}, J_i \subset j(I)\}.$$

We define then

$$(29) \quad K_\tau(I) := \sum_{\mathbf{J} \in \mathcal{J}(I)} \delta_\tau(\mathbf{J}).$$

With these notions we have the basic geometric estimate for the distortion function, in terms of the boundary homeomorphism:

**Theorem 2.6.** Let  $F : \mathbb{H} \rightarrow \mathbb{H}$  be the extension of a 1-periodic homeomorphism  $h : \mathbb{R} \rightarrow \mathbb{R}$ . Then for each  $I \in \mathcal{D}$

$$(30) \quad \sup_{z \in C_I} K(z, F) \leq C_0 K_\tau(I),$$

with a universal constant  $C_0$ .

**Proof.** The distortion properties of the Beurling-Ahlfors extension are well studied in the existing literature, but none of these works gives directly Theorem 2.6 as the main point for us is the linear dependence on the local distortion  $K_\tau(I)$ . The most elementary extension operator is due to Jerison and Kenig [14], see also [4, Section 5.8], but for this extension the linear dependence fails.

For the reader's convenience we sketch a proof for the theorem. We will modify the approach of Reed [25], and start with the simple Lemma.

**Lemma 2.7.** For each dyadic interval  $I = [k2^{-n}, (k+1)2^{-n}]$ , with left half  $I_1 = [k2^{-n}, (k+1/2)2^{-n}]$  and right half  $I_2 = I \setminus I_1$ , we have

$$\frac{1}{1 + \delta_\tau(I)} |\tau(I)| \leq |\tau(I_1)|, |\tau(I_2)| \leq \frac{\delta_\tau(I)}{1 + \delta_\tau(I)} |\tau(I)|$$

with

$$\frac{1}{|I|} \int_I h(t) - h(k2^{-n}) dt \leq \frac{3\delta_\tau(I)}{1 + 3\delta_\tau(I)} |\tau(I)|$$

and

$$\frac{1}{|I|} \int_I h((k+1)2^{-n}) - h(t) dt \leq \frac{3\delta_\tau(I)}{1 + 3\delta_\tau(I)} |\tau(I)|$$

**Proof.** The definition of  $\delta_\tau(I)$  gives directly the first estimate. As  $h(t) \leq h((k+1/2)2^{-n})$  on the left half and  $h(t) \leq h((k+1)2^{-n})$  on the right half of  $I$ ,

$$\frac{1}{|I|} \int_I h(t) - h(k2^{-n}) dt \leq \left( \frac{1}{2} \frac{\delta_\tau(I)}{1 + \delta_\tau(I)} + \frac{1}{2} \right) |\tau(I)| \leq \frac{3\delta_\tau(I)}{1 + 3\delta_\tau(I)} |\tau(I)|.$$

The last estimate follows similarly.  $\square$

To continue with the proof of Theorem 2.6, the pointwise distortion of the extension  $F$  is easy to calculate explicitly, and we obtain [6, 25] the following estimate, sharp up to a multiplicative constant,

$$(31) \quad K(x + iy, F) \leq \left( \frac{\alpha(x, y)}{\beta(x, y)} + \frac{\beta(x, y)}{\alpha(x, y)} \right) \left[ \frac{\tilde{\alpha}(x, y)}{\alpha(x, y)} + \frac{\tilde{\beta}(x, y)}{\beta(x, y)} \right]^{-1},$$

where

$$\alpha(x, y) = h(x + y) - h(x), \quad \beta(x, y) = h(x) - h(x - y)$$

and

$$\tilde{\alpha}(x, y) = h(x + y) - \frac{1}{y} \int_x^{x+y} h(t) dt, \quad \tilde{\beta}(x, y) = \frac{1}{y} \int_{x-y}^x h(t) dt - h(x - y).$$

Now the argument of Reed [25, pp. 461-464], combined with Lemma 2.7 and its estimates, precisely shows that  $K(x + iy, F) \leq 24 \max \delta_\tau(\tilde{I})$ , where  $\tilde{I}$  runs over the intervals with endpoints contained in the set

$$(32) \quad \{x, x \pm y/4, x \pm y/2, x \pm y\}.$$

Thus, for example, if we fix  $k \in \mathbb{Z}$  and  $n \in \mathbb{N}$ , we get for the corner point  $z = k2^{-n} + i2^{-n}$  of the Whitney cube  $C_I$  the estimate

$$(33) \quad K(k2^{-n} + i2^{-n}, F) \leq 24 \sum_{\mathbf{J}} \delta_\tau(\mathbf{J}), \quad \mathbf{J} = (J_1, J_2), \quad J_i \in \mathcal{D}_{n+3},$$

where  $J_i \subset j(I)$  as above. For a general point  $z = x + iy \in C_I$ , we have to take a few more generations of dyadic intervals. Here  $[x, x + y/4]$  has length at least  $2^{-n-3}$ . On the other hand, for any (non-dyadic) interval  $\tilde{I}$  with  $2^{-m} \leq |\tilde{I}| < 2^{-m+1}$ , one observes that it contains a dyadic interval of length  $2^{-m-1}$  and is contained inside a union of at most three dyadic intervals of length  $2^{-m}$ . By this manner one estimates

$$\delta_\tau(\tilde{I}) \leq \sum_{\tilde{\mathbf{J}}} \delta_\tau(\mathbf{J}), \quad \text{where } \mathbf{J} = (J_1, J_2), \quad J_i \in \mathcal{D}_{m+2} \quad \text{and} \quad J_i \cap \tilde{I} \neq \emptyset.$$

Choosing the endpoints of  $\tilde{I}$  from the set in (32) then gives the bound (30). Note that the estimates hold also for  $n = 0$ , since by (21) we have  $K(z, F) \leq 5/4$  whenever  $y \geq 1$ . Hence the proof of Theorem 2.6 is complete.  $\square$

### 3. EXPONENTIAL OF GFF AND RANDOM HOMEOMORPHISMS OF $\mathbb{T}$

**3.1. Trace of the Gaussian Free Field.** Let us recall that the 2-dimensional Gaussian Free Field (in other words, the massless free field)  $Y$  in the plane has the covariance

$$\mathbb{E} Y(x) Y(x') = \log \left( \frac{1}{|x - x'|} \right), \quad x, x' \in \mathbb{R}^2.$$

Actually, the definition of this field in the whole plane has to be done carefully, because of the blowup of the logarithm at infinity. However, the definition of the trace  $X := Y|_{\mathbb{T}}$  on the unit circle  $\mathbb{T}$  avoids this problem, since it is formally obtained by requiring (in the convenient complex notation)

$$(34) \quad \mathbb{E} X(z) X(z') = \log \left( \frac{1}{|z - z'|} \right), \quad z, z' \in \mathbb{T}.$$

The above definition needs to be made precise. In order to serve also readers with less background in non-smooth stochastic fields, let us first recall the definition of Gaussian random variables with values in the space of distributions  $\mathcal{D}'(\mathbb{T})$ . Recall first that an

element in  $F \in \mathcal{D}'(\mathbb{T})$  is real-valued if it takes real values on real-valued test functions. Identifying  $\mathbb{T}$  with  $[0, 1)$  a real-valued  $F$  may be written as

$$F = a_0 + \sum_{n=1}^{\infty} (a_n \cos(2\pi nt) + b_n \sin(2\pi nt)),$$

with real coefficients satisfying  $|a_n|, |b_n| = O(n^a)$  for some  $a \in \mathbb{R}$ . Conversely, every such Fourier series converges in  $\mathcal{D}'(\mathbb{T})$ .

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  stand for a probability space. A map  $X : \Omega \rightarrow \mathcal{D}'(\mathbb{T})$  is a (real-valued) centered  $\mathcal{D}'(\mathbb{T})$ -valued Gaussian if for every (real-valued)  $\psi \in C_0^\infty(\mathbb{T})$  the map

$$\omega \mapsto \langle X(\omega), \psi \rangle$$

is a centered Gaussian on  $\Omega$ . Here  $\langle \cdot, \cdot \rangle$  refers to the standard distributional duality. Alternatively, one may define such a random variable by requiring that a.s.

$$X(\omega) = A_0(\omega) + \sum_{n=1}^{\infty} (A_n(\omega) \cos(2\pi nt) + B_n(\omega) \sin(2\pi nt))$$

where the  $A_n, B_n$  are centered Gaussians satisfying  $\mathbb{E} A_n^2, \mathbb{E} B_n^2 = O(n^a)$  for some  $a \in \mathbb{R}$ . The random variable  $X$  is stationary if and only if the coefficients  $A_0, A_1, \dots, B_1, B_2, \dots$  are independent.

Due to Gaussianity, the distribution of  $X$  is uniquely determined by the knowledge of the covariance operator  $C_X : C^\infty(\mathbb{T}) \rightarrow \mathcal{D}'(\mathbb{T})$ , where

$$\langle C_X \psi_1, \psi_2 \rangle := \mathbb{E} \langle X(\omega), \psi_1 \rangle \langle X(\omega), \psi_2 \rangle.$$

In case the covariance operator has an integral kernel we use the same symbol for the kernel, and in this case for almost every  $z \in \mathbb{T}$  one has

$$(C_X \psi)(z) = \int_{\mathbb{T}} C_X(z, w) \psi(w) m(dw),$$

where  $m$  stands for the normalized Lebesgue measure on  $\mathbb{T}$ . Most of the above definitions and statements carry directly on  $\mathcal{S}'(\mathbb{R})$ -valued random variables, but the above knowledge is enough for our purposes.

The exact definition of (34) is understood in the above sense:

**Definition 3.1.** *The trace  $X$  of the 2 dimensional GFF on  $\mathbb{T}$  is a centered  $\mathcal{D}'(\mathbb{T})$ -valued Gaussian random variable such that its covariance operator has the integral kernel*

$$C_X(z, z') = \log \left( \frac{1}{|z - z'|} \right), \quad z, z' \in \mathbb{T}.$$

Observe that in the identification  $\mathbb{T} = [0, 1)$  the covariance of  $X$  takes the form

$$(35) \quad C_X(t, u) = \log \left( \frac{1}{2 \sin(\pi |t - u|)} \right) \quad \text{for } t, u \in [0, 1).$$

The existence of such a field is most easily established by writing down the Fourier expansion:

$$(36) \quad X = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (A_n \cos(2\pi nt) + B_n \sin(2\pi nt)), \quad t \in [0, 1),$$

where all the coefficients  $A_n \sim N(0, 1) \sim B_n$  ( $n \geq 1$ ) are independent standard Gaussians. Writing  $X$  as  $\sum_n \frac{1}{\sqrt{n}}(\alpha_n z^n + \bar{\alpha}_n \bar{z}^n)$  with  $|z| = 1$  and  $\alpha = \frac{1}{2}(A + iB)$  it is readily checked that it has the stated covariance.

What makes the trace  $X$  of the 2 dimensional GFF particularly natural for the circle homeomorphisms are its invariance properties, that  $X$  is Möbius invariant *modulo constants*. To see this note that the covariance  $C(z, z') = \log(1/|z - z'|)$  satisfies the transformation rule

$$C(g(z), g(z')) = C(z, z') + A(z) + B(z'),$$

where  $A$  (resp.  $B$ ) is independent of  $z'$  (resp.  $z$ ), whence the last two terms vanish in integration against mean zero test-functions.

It is well-known that with probability one  $X(\omega)$  is not an element in  $L^1(\mathbb{T})$ , (or a measure on  $\mathbb{T}$ ), but it just barely fails to be a function valued field. Namely, if  $\varepsilon > 0$  and one considers the  $\varepsilon$ -smoothened field  $(1 - \Delta)^{-\varepsilon} X$ , one computes that this field has a Hölder-continuous covariance, whence its realization belongs to  $C(\mathbb{T})$  almost surely. This follows from the following fundamental result of Dudley that we will use repeatedly below.

**Theorem 3.2.** Let  $(Y_t)_{t \in T}$  be a centered Gaussian field indexed by the set  $T$ , where  $T$  is a compact metric space with distance  $d$ . Define the (pseudo)distance  $d'$  on  $T$  by setting  $d'(t_1, t_2) = (\mathbb{E}|Y_{t_1} - Y_{t_2}|^2)^{1/2}$  for  $t_1, t_2 \in T$ . Assume that  $d' : T \times T \rightarrow \mathbb{R}$  is continuous. For  $\delta > 0$  denote by  $N(\delta)$  the minimal number of balls of radius  $\delta$  in the  $d'$ -metric needed to cover  $T$ . If

$$(37) \quad \int_0^1 \sqrt{\log N(\delta)} d\delta < \infty,$$

then  $Y$  has a continuous version, i.e. almost surely the map  $T \ni t \mapsto Y_t$  is continuous.

For a proof we refer to [2, Thm 1.3.5] or [17, Thm 4, Chapter 15]. The second result we will need is the Borell-TIS inequality (due to C. Borell, or, independently, B. Tsirelson, I. Ibragimov and V. Sudakov). According to the inequality, the tail of the supremum is dominated by a Gaussian tail:

$$(38) \quad \mathbb{P}(\sup_{t \in T} |Y_t| > u) \leq A \exp(Bu - u^2/2\sigma_T^2),$$

where  $\sigma_T := \max_{t \in T} (\mathbb{E} Y_t^2)^{1/2}$ , and the constants  $A$  and  $B$  depend on  $(T, d')$ , see [2, Section 2.1]. We shall also need an explicit quantitative version of this inequality in the special case where  $T$  is an interval:

**Lemma 3.3.** Let  $T = [x_0, x_0 + \ell]$ , and suppose that the covariance is Lipschitz continuous with constant  $L$ , i.e.  $\mathbb{E}|Y_t - Y_{t'}|^2 \leq L|t - t'|$  for  $t, t' \in T$ . Assume also that  $Y_{t_0} \equiv 0$  for a  $t_0 \in T$ . Then

$$\mathbb{P}(\sup_{t \in T} |Y_t| > \sqrt{L\ell}u) \leq c(1 + u)e^{-u^2/2},$$

where  $c$  is a universal constant.

**Proof.** The result is essentially due to Samorodnitsky [27] and Talagrand [31]. It is a direct consequence of [2, Thm 4.1.2] since after scaling it is possible to assume that  $L = 1 = \ell$ , and then  $\sigma_T \leq 1$  and  $N(\varepsilon) \leq 1/\varepsilon^2$ .  $\square$

**3.2. White noise expansion.** The Fourier series expansion (36) is often not the most suitable representation of  $X$  for explicit calculations. Instead, we shall apply a representation that uses white noise in the upper half plane, due to Bacry and Muzy [5]. The white noise representation is very convenient since it allows one to consider correlation between different scales both in the stochastic side and on  $\mathbb{T}$  in a flexible and geometrically transparent manner. Moreover, as we define the exponential of the field  $X$  in the next subsection we are then able to refer to known results in [5] and elsewhere.

To commence with, let  $\lambda$  stand for the hyperbolic area measure in the upper half plane  $\mathbb{H}$ ,

$$\lambda(dxdy) = \frac{dxdy}{y^2}.$$

Denote by  $w$  a white noise in  $\mathbb{H}$  with respect to measure  $\lambda$ . More precisely,  $w$  is a centered Gaussian process indexed by Borel sets  $A \in \mathcal{B}_f(\mathbb{H})$ , where

$$\mathcal{B}_f(\mathbb{H}) := \{A \subset \mathbb{H} \text{ Borel} \mid \lambda(A) < \infty \text{ and } \sup_{(x,y),(x',y') \in A} |x' - x| < \infty\},$$

i.e. Borel sets of finite hyperbolic area and finite width, and with the covariance structure

$$\mathbb{E}(w(A_1)w(A_2)) = \lambda(A_1 \cap A_2), \quad A_1, A_2 \in \mathcal{B}_f(\mathbb{H}).$$

We shall need a periodic version of  $w$ , which can be identified with a white noise on  $\mathbb{T} \times \mathbb{R}_+$ . Thus, define  $W$  as the centered Gaussian process, also indexed by  $\mathcal{B}_f(\mathbb{H})$ , and with covariance

$$\mathbb{E}(W(A_1)W(A_2)) = \lambda\left(A_1 \cap \bigcup_{n \in \mathbb{Z}} (A_2 + n)\right).$$

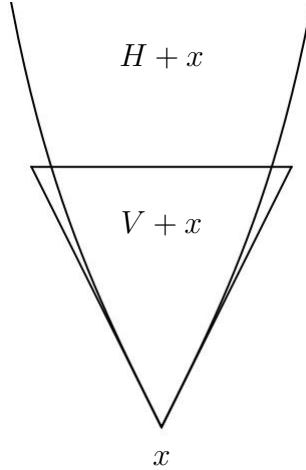


FIGURE 1. White noise dependence of the fields  $H(x)$  and  $V(x)$

We will represent the trace  $X$  using the following random field  $H(x)$ . Consider the wedge shaped region

$$H := \{(x, y) \in \mathbb{H} : -1/2 < x < 1/2, \ y > \frac{2}{\pi} \tan(|\pi x|)\}$$

and formally set

$$H(x) := W(H + x), \quad x \in \mathbb{R}/\mathbb{Z},$$

see Fig. 1. The reader should think about the  $y$  axis as parametrizing the spatial scale. Roughly, the white noise at level  $y$  contributes to  $H(x)$  in that spatial scale.

To define  $H$  rigorously we introduce a short distance cutoff parameter  $\varepsilon > 0$  and, given any  $A \in \mathcal{B}_f(\mathbb{H})$ , let  $A_\varepsilon := A \cap \{y > \varepsilon\}$ . Then set

$$(39) \quad H_\varepsilon(x) := W(H_\varepsilon + x).$$

According to Dudley's Theorem 3.2 one may pick a version of the white noise  $W$  in such a way that the map

$$(0, 1) \times \mathbb{R} \supset (\varepsilon, x) \mapsto H_\varepsilon(x)$$

is continuous. In the limit  $\varepsilon \rightarrow 0^+$  we nicely recover  $X$ :

**Lemma 3.4.** *One may assume that the version of the white noise is chosen so that for any  $\varepsilon > 0$  the map  $x \mapsto H_\varepsilon(x)$  is continuous, and as  $\varepsilon \rightarrow 0^+$  it converges in  $\mathcal{D}'(\mathbb{T})$  to a random field  $H$ . Moreover*

$$H \sim X + G,$$

where  $G \sim N(0, 2 \log 2)$  is a (scalar) Gaussian factor, independent of  $X$ .

**Proof.** Observe first that we may compute formally (as  $H(\cdot)$  is not well-defined pointwise) for  $t \in (0, 1)$

$$\mathbb{E} H(0)H(t) = \lambda(H \cap (H + t)) + \lambda(H \cap (H + t - 1)).$$

The first term in the right hand side can be computed as follows:

$$\begin{aligned} \lambda(H \cap (H + t)) &= 2 \int_{t/2}^{1/2} \left( \int_{(2/\pi) \tan(\pi x)}^{\infty} \frac{dy}{y^2} \right) dx = \pi \int_{t/2}^{1/2} \cot(\pi x) dx \\ &= \log \frac{1}{\sin(\pi t/2)}. \end{aligned}$$

Hence we obtain by symmetry

$$\begin{aligned} (40) \quad \mathbb{E} H(0)H(t) &= \log \left( \frac{1}{\sin(\pi t/2)} \right) + \log \left( \frac{1}{\sin(\pi(1-t)/2)} \right) \\ &= 2 \log 2 + \log \left( \frac{1}{2 \sin(\pi t)} \right). \end{aligned}$$

The stated relation between  $X$  and  $H$  follows immediately from this as soon as we prove the rest of the theorem. Observe that the covariance of the smooth field  $H_\varepsilon(\cdot)$  on  $\mathbb{T}$  converges to the above pointwise for  $t \neq 0$ . A computation shows that for any  $\delta > 0$  the covariance of the field

$$[0, 1] \times [0, 1] \supset (\varepsilon, x) \mapsto (1 - \Delta)^{-\delta} H_\varepsilon(x) := H_{\varepsilon, \delta}(x)$$

(at  $\varepsilon = 0$  one applies the covariance computed in (40)) is Hölder-continuous in the compact set  $[0, 1] \times \mathbb{T}$ , whence Dudley's theorem yields the existence of a continuous version on that set, especially  $H_{\varepsilon, \delta}(\cdot) \rightarrow H_{0, \delta}(\cdot)$  in  $C(\mathbb{T})$ , hence in  $\mathcal{D}'(\mathbb{T})$ . By applying  $(1 - \Delta)^\delta$  on both sides we obtain the stated convergence. Especially, we see that the convergence takes place in any of the Zygmund spaces  $C^{-\delta}(\mathbb{T})$ , with  $\delta > 0$ .  $\square$

The logarithmic singularity in the covariance of  $H(x)$  is produced by the asymptotic shape of the region  $H$  near the real axis. It will be often convenient to work with the following auxiliary field, which is geometrically slightly easier to tackle while for small scales it does not distinguish between  $w$  and its periodic counterpart  $W$ . Thus, consider this time the triangular set

$$(41) \quad V := \{(x, y) \in \mathbb{H} : -1/4 < x < 1/4, \ 2|x| < y < 1/2\}.$$

and let  $V_\varepsilon(x) = W(V_\varepsilon + x)$  (see Fig 1.). The existence of the limit  $V(x) := \lim_{\varepsilon \rightarrow 0^+} V_\varepsilon(\cdot)$  is established just like for  $H$  and we get the covariance

$$(42) \quad \mathbb{E} V(x)V(x') = \log\left(\frac{1}{2|x-x'|}\right) + 2|x-x'| - 1$$

for  $|x-x'| \leq 1/2$  (while for  $|x-x'| > 1/2$  the periodicity must be taken into account).

Since the regions  $H$  and  $V$  have the same slope at the real axis the difference  $H(\cdot) - V(\cdot)$  is a quite regular field:

**Lemma 3.5.** *Denote  $\xi := \sup_{x \in [0,1], \varepsilon \in (0,1/2]} |V_\varepsilon(x) - H_\varepsilon(x)|$ . Then almost surely  $\xi < \infty$ . Moreover,  $\mathbb{E} \exp(a\xi) < \infty$  for all  $a > 0$ .*

**Proof.** We may write for  $\varepsilon \in [0, 1/2]$

$$V_\varepsilon(x) - H_\varepsilon(x) = T_\varepsilon(x) - G(x),$$

where  $T_\varepsilon(x)$  and  $G(x)$  are constructed as  $V_\varepsilon(x)$  out of the sets

$$G := H \cap \{y \geq 1/2\}, \quad T := V \setminus (H \cap \{y < 1/2\}).$$

Observe first that  $G(x)$  is independent of  $\varepsilon$  and it clearly has a Lipschitz covariance in  $x$ , so by Dudley's theorem and (38) almost surely the map  $G(\cdot) \in C(\mathbb{T})$  and, moreover, the tail of  $\|G(\cdot)\|_{C(\mathbb{T})}$  is dominated by a Gaussian, whence it's exponential moments are finite.

In a similar manner, the exponential integrability of  $\sup_{x \in [0,1], \varepsilon \in [0,1]} |T_\varepsilon(x)|$  is deduced from Dudley's theorem and (38) as soon as we verify that there is an exponent  $\alpha > 0$  such that for any  $|x-x'| \leq 1/2$  we have

$$(43) \quad \mathbb{E} |T_\varepsilon(x) - T_{\varepsilon'}(x')|^2 \leq c(|x-x'| + |\varepsilon - \varepsilon'|)^\alpha.$$

In order to verify this it is enough to change one variable at a time. Observe first that if  $1 > \varepsilon > \varepsilon' \geq 0$

$$\mathbb{E} |T_\varepsilon(x) - T_{\varepsilon'}(x)|^2 = \lambda(T \cap (\varepsilon' < y < \varepsilon)) \leq \int_{\varepsilon'}^{\varepsilon} cx^3 \leq c'|\varepsilon' - \varepsilon|,$$

where we applied the inequality  $0 \leq (t/2) - (1/\pi) \arctan(\pi t/2) \leq 2t^3$ .

Next we estimate the dependence on  $x$ . Denote  $z := |x-x'| \leq 1/2$ . We note that for any  $y_0 \in (0, 1/2)$  the linear measure of the intersection  $\{y = y_0\} \cap (T\Delta(T+z))$  is bounded by  $\min(2z, 4y_0^3)$ . Hence, by the definition of  $T_\varepsilon$  and the fact that for  $z = |x-x'| \leq 1/2$  the periodicity of  $W$  has no effect on estimating  $T$ , we obtain

$$\begin{aligned} \mathbb{E} |T_\varepsilon(x) - T_\varepsilon(x')|^2 &\leq \mathbb{E} |T_0(x) - T_0(x')|^2 = \lambda(T\Delta(T+z)) \\ &\leq 2z \int_{z^{1/3}}^{1/2} \frac{dy}{y^2} + \int_0^{z^{1/3}} \frac{4y^3}{y^2} \leq cz^{2/3}, \end{aligned}$$

which finishes the proof of the lemma.  $\square$



**3.3. Exponential of  $X$  and the random homeomorphism  $h$ .** We are now ready to define the exponential of the free field discussed in the Introduction and use it to define the random circle homeomorphisms.

By stationarity, the covariance

$$\gamma_H(\varepsilon) := \text{Cov}(H_\varepsilon(x)) = \mathbb{E}|H_\varepsilon(x)|^2$$

is independent of  $x$ , as is the quantity  $\gamma_V(\varepsilon)$  defined analogously. Fix  $\beta > 0$  (this parameter could be thought as an "inverse temperature"). Directly from definitions, for any  $x$  and for any bounded Borel-function  $g$  on  $[0,1)$  the processes

$$(44) \quad \varepsilon \mapsto \exp(\beta H_\varepsilon(x) - (\beta^2/2)\gamma_H(\varepsilon)) \quad \text{and}$$

$$(45) \quad \varepsilon \mapsto \int_0^1 \exp(\beta H_\varepsilon(u) - (\beta^2/2)\gamma_H(\varepsilon)) g(u) du$$

are  $L^1$ -martingales with respect to *decreasing*  $\varepsilon \in (0, 1/2]$ , whence they converge almost surely. Especially, the  $L^1$ -norm stays bounded and the Fourier-coefficients of the density  $\exp(\beta H_\varepsilon(x) - (\beta^2/2)\gamma_H(\varepsilon))$  converge as  $\varepsilon \rightarrow 0^+$ .

Now comparing these expressions with (2) and Lemma 3.4 we are led to the exact definition of our desired exponential

$${}^{\text{''}} d\tau = e^{\beta X(z)} dz {}^{\text{''}}.$$

Indeed, by the weak\*-compactness of the set of bounded positive measures we have the existence of the almost sure limit measure<sup>1</sup>

$$(46) \quad \text{a.s.} \quad \lim_{\varepsilon \rightarrow 0^+} e^{[\beta H_\varepsilon(x) - (\beta^2/2)\gamma_H(\varepsilon)]} e^{-\beta G} dx / 2^{\beta^2} =: \tau(dx) \quad \text{w}^* \text{ in } \mathcal{M}(\mathbb{T}),$$

where  $\mathcal{M}(\mathbb{T})$  stands for bounded Borel measures on  $\mathbb{T}$  and  $G \sim N(0, 2 \log 2)$  is a Gaussian (scalar) random variable.

In a similar manner one deduces the existence of the almost sure limit

$$(47) \quad \lim_{\varepsilon \rightarrow 0^+} \exp(\beta V_\varepsilon(x) - (\beta^2/2)\gamma_V(\varepsilon)) dx \stackrel{\text{w}^*}{=} \nu(dx)$$

Lemma 3.5 and stationarity yield immediately

**Lemma 3.6.** *There are versions of  $\tau$  and  $\nu$  on a common probability space, together with an almost surely finite and positive random variable  $G_1$ , with  $\mathbb{E} G_1^a < \infty$  for all  $a \in \mathbb{R}$ , so that for all Borel sets  $B$  one has*

$$\frac{1}{G_1} \tau(B) \leq \nu(B) \leq G_1 \tau(B).$$

Observe that the random variable  $G_1$  is independent of the set  $B$ . Thus, the measures are a.s. comparable.

Limit measures of above type, i.e. measures that are obtained as martingale limits of products (discrete, or continuous as in our case) of exponentials of independent Gaussian fields have been extensively studied in the literature. The study of "multiplicative chaos" starts with Kolmogorov, various versions of multiplicative cascade

<sup>1</sup>Observe that the limit measure is weak\*-measurable in the sense that for any  $f \in C(\mathbb{T})$  the integral  $\int_{\mathbb{T}} f(t) \tau(dt)$  is a well-defined random variable. In this paper all our random measures on  $\mathbb{T}$  are measurable (i.e. they are measure-valued random variables) in this sense. A simple limiting argument then shows that e.g.  $\tau(I)$  is a random variable for any interval  $I \subset \mathbb{T}$ .

models were advocated by Mandelbrot [22] and others, and Kahane made fundamental contributions to the rigorous mathematical theory, see [16],[18], [19]. We shall make use of these works, and [5], [26] in particular, which study in detail random measures closely related to our  $\nu$ .

For us the key points in constructing and understanding the random circle homeomorphism are the following properties of the measure  $\tau$  and its variant  $\nu$ .

**Theorem 3.7.** *Assume that  $\beta < \sqrt{2}$ .*

(i) *There are  $a = a(\beta) > 0$  and an a.s. finite random constant  $c = c(\omega, \beta)$  such that for all subintervals  $I \subset [0, 1)$  it holds*

$$0 < \tau(I) \leq c(\omega, \beta)|I|^a.$$

*Epecially,  $\tau$  is non-atomic.*

(ii) *For any subinterval  $I \subset [0, 1)$  the measure  $\tau$  satisfies*

$$(48) \quad \tau(I) \in L^p(\omega), \quad p \in (-\infty, 2/\beta^2).$$

*Moreover, if  $p \in (1, 2/\beta^2)$ , then*

$$(49) \quad \mathbb{E} \tau(I)^p < c(\beta, p)|I|^{\zeta_p}, \quad \text{with } \zeta_p > 1.$$

(iii) *One can replace  $\tau$  by the measure  $\nu$  in the statements (i) and (ii).*

**Proof.** We shall make use of one more auxiliary field, which (together with its exponential) is described in detail in [5]<sup>2</sup>. Define

$$U := \{(x, y) \in \mathbb{H} : -1/2 < x < 1/2, \ 2|x| < y\}.$$

and for  $x \in \mathbb{R}$ , let  $U(x) = w(U + x)$ . Here note in particular, that  $w$  is the nonperiodic white noise.

The covariance of  $U(\cdot)$  is easily computed (see [5, (25), p. 458]), and we obtain

$$(50) \quad \mathbb{E} U(x)U(x') = \log\left(\frac{1}{\min(y, 1)}\right) \quad \text{where } y := |x - x'|.$$

As before define the cutoff field  $U_\varepsilon(x) = w(U_\varepsilon + x)$ . Then  $U_\varepsilon$  is (locally) very close to our field  $V_\varepsilon(\cdot)$ . Indeed, let  $I$  be an interval of length  $|I| = \frac{1}{2}$ . Then  $V(\cdot)|_I$  is equal in law with  $w(\cdot + V)|_I$  since the periodicity of the white noise  $W$  will not enter. Thus we may realize  $U_\varepsilon|_I$  and  $V_\varepsilon|_I$  for  $\varepsilon \in (0, 1/2)$  in the same probability space so that

$$U_\varepsilon - V_\varepsilon := Z = w(x + U \cap \{y > 1/2\}).$$

We may again apply Dudley's theorem and eq. (38) to the random variable

$$(51) \quad \xi_1 := \sup_{x \in I, \varepsilon \in (0, 1/2]} |V_\varepsilon(x) - U_\varepsilon(x)| < \infty \quad \text{almost surely.}$$

Moreover,  $\mathbb{E} \exp(a\xi_1) < \infty$  for all  $a > 0$ . By denoting  $G_2 := \exp(a\xi_1)$  we thus have an analogue of Lemma 3.6

$$(52) \quad \frac{1}{G_2} \tau(B) \leq \nu(B) \leq G_2 \tau(B),$$

---

<sup>2</sup> $U_0$  corresponds to the simple case of log-normal MRM, see [5, p. 462, (28)], and  $T = 1$  in [5, p. 455, (15)].

for all  $B \subset I$ , and the auxiliary variable  $G_2$  satisfies  $\mathbb{E} G_2^p < \infty$  for all  $p \in \mathbb{R}$ . As an aside, note that we cannot have (52) for the full interval  $I = [0, 1]$ , as  $V$  is 1-periodic while  $U$  is not.

In a similar manner as for the measures  $\tau$  and  $\nu$  one deduces the existence of the almost sure limit

$$(53) \quad \lim_{\varepsilon \rightarrow 0^+} \exp(\beta U_\varepsilon(x) - (\beta^2/2)\gamma_U(\varepsilon)) dx =: \eta(dx),$$

where the limit takes place locally weak\* on the space of locally finite Borel-measures on the real-axis.

Now for proving the theorem, by (51) and Lemma 3.5 it is enough to check the corresponding claims (i) and (ii) for the random measure  $\eta$ , as one may clearly assume that  $|I| \leq 1/2$ . We start with claim (ii), which in the case of positive moments  $p > 0$  is due to Kahane (see [19],[16]). Bacry and Muzy [5, Appendix D] give a nice proof by adapting the argument of Kahane and Peyriere [19] (who considered a cascade model) to cover the measure  $\eta$ .

Finiteness of negative moments is announced in [26, Prop. 3.5], where it is stated that the argument given by Molchan [23] in the case of the cascade model carries through. For the readers convenience, we include the details for the negative moments in an Appendix.

Fact (49) for  $\eta$  is [5, Theorem 4] where it is observed that one may take  $\zeta_p = p - \beta^2(p^2 - p)/2$ . In order to treat (i), choose  $p \in (1, 2/\beta^2)$  and let  $a > 0$  be so small that  $b := \zeta_p - pa > 1$ . Chebychev's inequality in combination with (49) yields that  $\mathbb{P}(\eta(I) > |I|^a) \lesssim |I|^b$ . In particular,  $\sum_I \mathbb{P}(\eta(I) > |I|^a) < \infty$ , where one sums over dyadic subintervals of  $[0, 1]$ . The same holds true if one sums over the same dyadic subintervals shifted by their half-length. This observation in combination with the Borel-Cantelli lemma yields the desired upper estimate for the measure  $\eta$ . This immediately implies that  $\eta$  is non-atomic.

Finally, in order to sketch a proof of the non-degeneracy of  $\eta$  over any subinterval, we partition the upper half plane into vertical strips and define

$$(54) \quad U(x, j) := w((U + x) \cap \{1/(j+1) < y < 1/j\}).$$

Let then

$$f_j(x) = \exp(\beta U(x, j) - (\beta^2/2)\gamma_U(j)),$$

where  $\gamma_U(j) = \text{Cov}(U(x, j))$ . We may now write  $\eta$  as the a.s. limit

$$\eta(dx) = \text{w}^* \lim_{k \rightarrow \infty} \left( \prod_{j=0}^k f_j(x, \omega) \right) dx,$$

where the densities  $f_j(x, \omega)$  are independent and a.s. bounded from below by a positive constant. Moreover,  $\mathbb{E} f_j(x) = 1$  for each  $x, j$ . Let  $I$  be a dyadic subinterval and denote  $Y_k(\omega) := \int_I \left( \prod_{j=1}^k f_j(x, \omega) \right) dx$ . By Kolmogorov's 0-1 law, probability for  $\lim_{k \rightarrow \infty} Y_k = 0$  is either zero or one. The first alternative can be ruled out by observing that  $\mathbb{E} Y_k = |I|$  for all  $k$  and that  $(Y_k)_{k \geq 1}$  is an  $L^p$ -martingale with  $p > 1$ , according to fact (ii). Let us finally remark that the non-degeneracy and non-atomic nature for  $\eta$  can also be found in [16] and [18], see also [5, Theorems 1 and 2].  $\square$

Note that the exact scaling law of the measure  $\eta$  we used in the above proof is given in [5, Thm. 4]. Indeed, for any  $\varepsilon, \lambda \in (0, 1)$  one has the equivalence of laws

$$U_{\varepsilon\lambda}(\lambda \cdot)|_{[0,1]} \sim G_\lambda + U_\varepsilon|_{[0,1]}$$

where  $G_\lambda \sim N(0, \log(1/\lambda))$  is a Gaussian independent of  $U$ . Therefore, one has the equivalence of laws for measures on  $[0, 1]$ :

$$(55) \quad \eta(\lambda \cdot) \sim \lambda e^{\beta G_\lambda + \log(\lambda)\beta^2/2} \eta$$

and hence scale invariance of the ratios

$$(56) \quad \frac{\eta([\lambda x, \lambda y])}{\eta([\lambda a, \lambda b])} \sim \frac{\eta([x, y])}{\eta([a, b])}.$$

In turn, the exact scaling law of  $\tau$  is best described in terms of Möbius transformations of the circle. We do not state it as we do not need it later on.

To finish this section we are now able to define our circle homeomorphism  $h$ .

**Definition 3.8.** Assume that  $\beta^2 < 2$ . The random homeomorphism  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  is obtained by setting

$$(57) \quad \phi(e^{2\pi i x}) = e^{2\pi i h(x)},$$

where we let

$$(58) \quad h(x) = h_\beta(x) = \tau([0, x]) / \tau([0, 1]) \quad \text{for } x \in [0, 1],$$

and extend periodically over  $\mathbb{R}$ .

**Remark 3.9.** Theorem 3.7 (i) and (ii) precisely contain what is needed to ensure that  $h$  is a Hölder continuous homeomorphism for  $\beta^2 < 2$ . As an aside, let us note that defining  $\tau_\varepsilon$  as in the LHS of eq. (46) the limit  $\lim_{\varepsilon \rightarrow 0} \tau_\varepsilon = 0$  for  $\beta^2 \geq 2$ . However, it is a natural conjecture that letting  $h_\varepsilon$  to be given by (58) with  $\tau$  replaced by  $\tau_\varepsilon$ , the limit for  $h_\varepsilon$  exists in a suitable sense as  $\varepsilon \rightarrow 0^+$  also for  $\beta^2 \geq 2$ . Indeed, the normalized measure in eq. (58) appears in the physics literature as the Gibbs measure of a Random Energy model for logarithmically correlated energies [11], [12], [13] and the  $\beta^2 > 2$  corresponds to a low temperature "spin glass" phase. However, we don't expect the limiting  $h$  to be continuous if  $\beta^2 > 2$ .

**Question 3.10.** Is  $\beta \rightarrow h_\beta(x)$  almost surely continuous?

#### 4. PROBABILISTIC ESTIMATES FOR LEHTO INTEGRALS

**4.1. Notation and statement of the main estimate.** We will now set to study the Lehto integral of eq. (11) for the random homeomorphism constructed in the previous section. As explained in Section 2.4, it suffices to work in the infinite strip  $S = \mathbb{R} \times [0, 2]$  where the extension  $F$  of the random homeomorphism  $h$  is non-trivial. We use the bound (30) for the (random) pointwise distortion  $K = K(z, F)$  of this extension, and hence it turns out convenient to define  $K_\tau$  in the upper half plane by setting

$$(59) \quad K_\tau(z) := K_\tau(I) \quad \text{whenever } z \in C_I.$$

A lower bound for the Lehto integral (11) is then obtained by replacing  $K$  there by  $K_\tau$ . We similarly define  $K_\nu(z)$  for  $z \in \mathbb{H}$ , via the modified Beurling - Ahlfors extension of the periodic homeomorphism defined by the measure  $\nu$ .

It turns out that we only need to control Lehto integrals centered at real axis and with some (arbitrarily small, but fixed) outer radius. For this purpose fix (large)  $p \in \mathbb{N}$  and choose  $\rho = 2^{-p}$ , where final choice of  $p$  will be done in Subsection 4.3 below.

Our main probabilistic estimate is the following result.

**Theorem 4.1.** *Let  $w_0 \in \mathbb{R}$  and let  $\beta < \sqrt{2}$ . Then there exists  $b > 0$  and  $\rho_0 > 0$  together with  $\delta(\rho) > 0$  such that for positive  $\rho < \rho_0$  and  $\delta < \delta(\rho)$  the Lehto integral satisfies the estimate*

$$(60) \quad \mathbb{P}(L_{K_\nu}(w_0, \rho^N, 2\rho) < N\delta) \leq \rho^{(1+b)N}.$$

Observe that the estimates in the Theorem are in terms of  $K_\nu$  instead of  $K_\tau$ , which is the majorant for the distortion of the extension of the actual homeomorphism. However, this discrepancy will easily be taken care later on in the proof of Theorem 5.1 using the bounds in Lemma 3.5. The proof of the Theorem will occupy most of the present section, i.e. Subsections 4.2–4.4 below. Finally, we consider the almost sure integrability of the distortion in Subsection 4.5.

We next fix the notation that will be used for the rest of the present section, and explain the philosophy behind part (i) of the theorem. Given  $w_0$  we may choose the dyadic intervals in Theorem 2.6. as  $w_0 + I$ . Then, by stationarity we may assume that  $w_0 = 0$ . Let  $S_r$  denote the circle of radius  $r > 0$  with center at the origin. Define (with slight abuse) for  $r \leq 2\rho$

$$(61) \quad K_\nu(r) := \sum_{I: C_I \cap S_r \neq \emptyset} |I| K_\nu(I)$$

and observe that

$$(62) \quad L_{K_\nu}(0, \rho^N, 2\rho) \geq c \sum_{n=1}^N M_n,$$

where

$$(63) \quad M_n = \int_{\rho^n}^{2\rho^n} \frac{dr}{K_\nu(r)}.$$

Thus, in order to prove part (i) of the Theorem it is enough to verify for  $\beta < \sqrt{2}$  that for small enough  $\rho > 0$  and  $0 < \delta < \delta(\rho)$  one has

$$(64) \quad \mathbb{P}\left(\sum_{n=1}^N M_n < N\delta\right) \leq \rho^{(1+b)N}.$$

If the summands  $M_j$  in (64) were independent, the estimate would follow easily from basic large deviation estimates. However, they are far from being independent. Nevertheless, by the geometry of the setup in the white noise upper half plane, one expects that there is some kind of exponential decay of dependence, but due to the complicated structure of the Lehto integrals we need to go through a non-trivial technical analysis in order to be able to get hold on the exponential decay.

**4.2. Correlation structure of the  $M_j$ 's.** In this Section we will study how the random variables  $M_n$  are correlated with each other. As one can easily gather from the representation of the field  $\nu$  in terms of the white noise, all of the variables  $M_n$  with  $n = 1, 2, \dots$  are correlated with each other. Our basic strategy is to estimate  $M_n$  from below by the quantity

$$M'_n = m_n s_n \sigma_n$$

(see (85) below), where the random variables  $m_n$  depend only the white noise on the scale  $\sim \rho^n$  and form an independent set. The variables  $s_n$  will provide an estimate of *upscale correlations*, i.e. the dependence of  $M'_n$  on the white noise over the larger spatial scales  $\{|x| \gtrsim \rho^{n-1}\}$ . In turn, the variables  $\sigma_n$  measure the *downscale correlations* that corresponds to the dependence of  $M'_n$  on white noise over  $\{|x| \lesssim \rho^{n+1}\}$ . It turns out that the downscale correlations are harder to estimate.

We start with the upscale correlations and introduce some terminology. For a Borel-measurable  $S \subset \mathbb{H}$  let  $\mathcal{B}_S$  be the  $\sigma$ -algebra generated by the randoms variables  $W(A)$ , where  $A$  runs over Borel-measurable subsets  $A \subset S$ . We will call a  $\mathcal{B}_S$  measurable random variable for short  $S$  measurable. Let

$$V_I := \cup_{x \in I} (V + x)$$

where we recall  $V$  is given by (41). Then  $\nu(I)/\nu(J)$  is  $V_{I \cup J}$  measurable and by (29) we see that  $K_\nu(I)$  is  $V_{j(I)}$  measurable (recall that  $j(I)$  denotes the union of  $I$  with its neighboring dyadic intervals). From (61) we deduce that  $M_n$  is  $V_{B_n}$  measurable where  $B_n := B(0, 4\rho^n)$ . Indeed, the Whitney cubes  $C_I$  that intersect the annulus  $A_n := B(0, 2\rho^n) \setminus B(0, \rho^n)$  have  $I \subset B(0, 2\rho^n)$  and thus  $j(I) \subset B(0, 4\rho^n)$ .

We now decompose  $V(\cdot)|_{B_n}$  to scales using the white noise. Denote in general for  $0 \leq \varepsilon < \varepsilon'$ ,

$$(65) \quad V(x, \varepsilon, \varepsilon') := W((V + x) \cap \{\varepsilon < y < \varepsilon'\}).$$

Set for  $n \geq 1$

$$(66) \quad \psi_n(x) = V(x, 0, \rho^{n-\frac{1}{2}})$$

and for  $k \geq 0$

$$(67) \quad \zeta_k(x) = V(x, \rho^{k+\frac{1}{2}}, \rho^{k-\frac{1}{2}}).$$

Denoting

$$(68) \quad \Lambda_n = \{z \in \mathbb{H} : y \leq \rho^{n-\frac{1}{2}}\}$$

we see that in any open set  $U$  the field  $\psi_n$  is  $(\bigcup_{y \in U} V_y) \cap \Lambda_n$  measurable. In a similar way,  $\zeta_k(x)$  is  $V_x \cap (\Lambda_k \setminus \Lambda_{k+1})$  measurable and since these regions are disjoint the field  $V$  decomposes to a sum of independent fields

$$(69) \quad V = \psi_n + \sum_{k=0}^{n-1} \zeta_k := \psi_n + z_n.$$

Let  $\nu_n$  be the measure defined as  $\nu$  but with  $V$  replaced by  $\psi_n$ . Inserting the second decomposition in (69) to the measure  $\nu$  we have, for any  $I, J \subset B_n$

$$(70) \quad \frac{\nu(I)}{\nu(J)} \leq \frac{\nu_n(I)}{\nu_n(J)} \cdot \frac{\sup_{x \in B_n} e^{\beta z_n(x)}}{\inf_{x \in B_n} e^{\beta z_n(x)}}.$$

The first decomposition in (69) then gives

$$(71) \quad \frac{\sup_{x \in B_n} e^{\beta z_n(x)}}{\inf_{x \in B_n} e^{\beta z_n(x)}} \leq e^{\sum_{k=0}^{n-1} t_{n,k}} := s_n^{-1}$$

where

$$(72) \quad t_{n,k} := \log \frac{\sup_{x \in B_n} e^{\beta \zeta_k(x)}}{\inf_{x \in B_n} e^{\beta \zeta_k(x)}}.$$

Thus if we let

$$(73) \quad \mathcal{M}_n = \int_{\rho^n}^{2\rho^n} \frac{dr}{K_{\nu_n}(r)}$$

we arrive to the following lower bound for  $M_n$ :

$$(74) \quad M_n \geq \mathcal{M}_n s_n.$$

This is the desired decoupling upscale. Note that the fields  $\zeta_k$  become more regular as  $k$  decreases. This will lead to the following Proposition:

**Proposition 4.2.** *The random variables  $t_{n,k}$  satisfy*

$$(75) \quad \mathbb{P}(t_{n,k} > u\rho^{(n-k)/2-1/4}) \leq ce^{-u^2/c}, \quad k = 0, \dots, n-1,$$

where  $c$  is independent on  $\rho$ ,  $n$  and  $k$ . Moreover,  $t_{n,k}$  and  $t_{n,k'}$  are independent if  $k \neq k'$ .

The proof of this proposition is postponed to Subsection 4.4 below.

The decoupling downscale is done to the random variables  $\mathcal{M}_n$  in (73). Obviously  $\mathcal{M}_n$  and  $\mathcal{M}_m$  are dependent. However, as in (61), most of the terms  $K_{n,I} := K_{\nu_n}(I)$  are independent of  $\mathcal{M}_m$  if  $m > n$ . The few which are not we will process further in a moment.

So let us first look at the dependence of the  $K_{n,I}$  on the white noise. For  $U \subset \mathbb{R}$  set  $V_U^n := V_U \cap \Lambda_n$ . Then  $K_{n,I}$  is  $V_{j(I)}^n$  measurable and  $\mathcal{M}_m$  is  $V_{B_m}^m$  measurable. Some drawing will convince the reader that if  $\text{dist}(j(I), 0)$  is not too small  $K_{n,I}$  and  $\mathcal{M}_m$  are independent for  $m > n$ . Indeed, consider the ball  $B'_n = B(0, 2\rho^{n+\frac{1}{2}})$  so that  $B_{n+1} \subset B'_n \subset B_n$ . The regions  $V_{B_n \setminus B'_n}^n$  are disjoint (see Figure 2). Thus the  $\sigma$ -algebras  $\mathcal{B}_{V_{B_n \setminus B'_n}^n \setminus V_{B'_n}^n}$  are independent from each other for  $n = 1, 2, \dots$

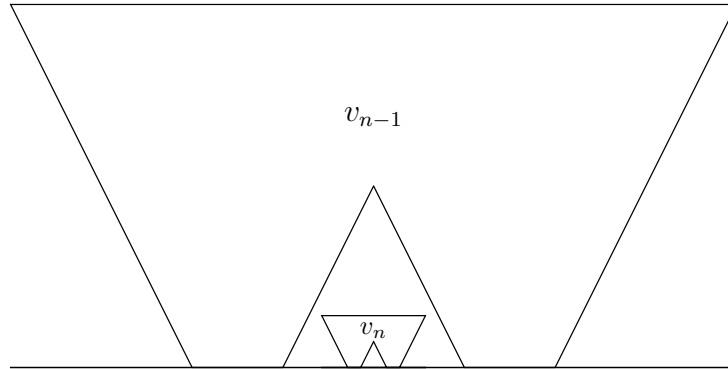


FIGURE 2. A schematic picture of the regions  $V_{B_n \setminus B'_n}^n := v_n$ , where the  $m_n$  are measurable

Let  $\mathcal{I}_n$  be the set of  $I \in \mathcal{D}$  such that the Whitney cube  $C_I$  intersects the annulus  $A(0, \rho^n, 2\rho^n)$  and  $j(I) \cap B'_n \neq \emptyset$  (some drawing shows such  $I \in \mathcal{D}_{np+i}$  for  $i = 0, \pm 1$ ). Moreover, for each fixed  $r \in (\rho^n, 2\rho^n)$  let  $\mathcal{I}_n(r)$  consist of those intervals  $I$  for which  $C_I \cap S_r \neq \emptyset$  and  $j(I) \cap B'_n = \emptyset$ . By (61) we then have

$$(76) \quad K_{\nu_n}(r) \leq \rho^n \left( \sum_{I \in \mathcal{I}_n} K_{n,I} + \sum_{I \in \mathcal{I}_n(r)} \rho^{-n} |I| K_{n,I} \right) := \rho^n (L_n + L_n(r)), \quad r \in (\rho^n, 2\rho^n).$$

Thus inserting (76) into (73) we get

$$(77) \quad \mathcal{M}_n \geq \int_{\rho^n}^{2\rho^n} \frac{1}{(L_n(r) + L_n)} \rho^{-n} dr.$$

The term  $L_n(r)$  in the integrand (77) is independent of  $\mathcal{M}_m$ ,  $m > n$ . However  $L_n$  is not and we will decouple it now. From (76) and (29) we get

$$(78) \quad L_n \leq \sum_{\mathbf{J}} \delta_{\nu_n}(\mathbf{J})$$

where the sum runs over a set of  $\mathbf{J} = (J_1, J_2)$  with  $J_i \in \cup_{i=0, \pm 1} \mathcal{D}_{np+5+i}$  and  $J_i \subset B_n$ . In particular

$$(79) \quad |J_i \setminus B'_n| \geq 2^{-np-7} = \rho^n 2^{-7}.$$

The sum in (78) has an  $n$ -independent number of terms (with multiplicities).

Next estimate  $\delta_{\nu_n}(\mathbf{J})$  in terms of a  $V_{B_n \setminus B'_n}^n$  measurable term and perturbation:

$$\begin{aligned} \delta_{\nu_n}(\mathbf{J}) &= \frac{\nu_n(J_1 \setminus B'_n) + \nu_n(J_1 \cap B'_n)}{\nu_n(J_2 \setminus B'_n) + \nu_n(J_2 \cap B'_n)} + (1 \leftrightarrow 2) \\ &\leq \frac{\nu_n(J_1 \setminus B'_n) + \nu_n(J_1 \cap B'_n)}{\nu_n(J_2 \setminus B'_n)} + (1 \leftrightarrow 2) \\ &= \delta_{\nu_n}(J_1 \setminus B'_n, J_2 \setminus B'_n) + \frac{\nu_n(J_1 \cap B'_n)}{\nu_n(J_2 \setminus B'_n)} + \frac{\nu_n(J_2 \cap B'_n)}{\nu_n(J_1 \setminus B'_n)}. \end{aligned}$$

Then decompose the perturbation further downscale:

$$\nu_n(J_i \cap B'_n) = \sum_{m=n+1}^{\infty} \nu_n(J_i \cap (B'_{m-1} \setminus B'_m))$$

and (recalling (28)) define

$$(80) \quad L_{n,n} = \sum_{(J_1, J_2) \in \mathcal{J}(I), I \in \mathcal{I}_n} \delta_{\nu_n}(J_1 \setminus B'_n, J_2 \setminus B'_n)$$

$$(81) \quad L_{n,m} = \sum_{(J_1, J_2) \in \mathcal{J}(I), I \in \mathcal{I}_n} \frac{\nu_n(J_1 \cap (B'_{m-1} \setminus B'_m))}{\nu_n(J_2 \setminus B'_n)} + (1 \leftrightarrow 2) \quad \text{for } m \geq n+1.$$

Then

$$L_n \leq \sum_{m=n}^{\infty} L_{n,m}.$$

Defining

$$(82) \quad m_n = \int_{\rho^n}^{2\rho^n} \frac{1}{(1 + L_n(r) + L_{n,n})} \rho^{-n} dr$$



and using the inequality

$$L_n(r) + L_n \leq (1 + L_n(r) + L_{n,n})(1 + \sum_{m=n+1}^{\infty} L_{n,m})$$

we get from (77)

$$(83) \quad \mathcal{M}_n \geq m_n \sigma_n$$

with

$$(84) \quad \sigma_n := (1 + \sum_{m=n+1}^{\infty} L_{n,m})^{-1}$$

Combining this with (74) we arrive to the desired bound of  $M_n$  in terms of random variables localized in the white noise:

$$(85) \quad M_n \geq m_n s_n \sigma_n := M'_n.$$

**Proposition 4.3. (i)** *The random variables  $m_n$  are  $V_{B_n \setminus B'_n}^n$  measurable,  $0 \leq m_n \leq 1$ , and they form an independent set. Moreover,*

$$\mathbb{P}(m_n \leq x) \leq Cx, \quad \text{for } x > 0,$$

where  $C$  is independent on  $\rho$  and  $n$ .

**(ii)** *There exists  $a > 0$ ,  $q > 1$  and  $C < \infty$  (independent of  $n, m$  and  $\rho$ ) such that for all  $m > n \geq 1$  the random variable  $L_{n,m}$  satisfies the estimate*

$$(86) \quad \mathbb{P}(L_{n,m} > \lambda) \leq C\lambda^{-q}\rho^{(m-n-1/2)(1+a)}.$$

Moreover,  $L_{n,m}$  is  $V_{B_n \setminus B'_m}^n$  measurable. Especially,  $L_{n,m}$  and  $L_{n',m'}$  are independent if  $n > m'$  or  $n' > m$ .

The proof of this Proposition is postponed to Subsection 4.4.

**4.3. Law of large numbers and proof of Theorem 4.1.** Here we prove our main probabilistic estimate assuming Propositions 4.2 and 4.3. By (85) we need to consider

$$(87) \quad P_N := \mathbb{P}(\sum_{n=1}^N M'_n < N\delta) = \mathbb{E} \chi(\sum_{n=1}^N m_n s_n \sigma_n \leq \delta N) =: \mathbb{E} \chi_{D_N},$$

where we denoted  $D_N := \{\omega : \sum_{n=1}^N m_n s_n \sigma_n \leq \delta N\}$ . For the sake of notational clarity we used above (and will often use later on) the shorthand  $\chi(A)$  for the indicator function  $\chi_A$ . In order to obtain the desired bound for  $P_N$  we insert suitable auxiliary characteristic functions in the expectation. Define

$$(88) \quad \chi_n := \prod_{m=n+1}^{\infty} \chi(L_{n,m} \leq 2^{n-m}\delta^{-1/4}) \prod_{m=0}^{n-1} \chi(t_{n,m} \leq 2^{m-n} \log(\frac{1}{2}\delta^{-1/4})) := \prod_{m \neq n} \chi_{n,m}.$$

On the support of  $\chi_n$  we have

$$\sum_{m=n+1}^{\infty} L_{n,m} \leq \delta^{-1/4}$$

and thus (for  $\delta < 1$  say)

$$\sigma_n \geq \frac{1}{2}\delta^{1/4}.$$

Similarly  $\sum_{m=0}^{n-1} t_{n,m} \leq \log \frac{1}{2}\delta^{-1/4}$  and so

$$s_n \geq 2\delta^{1/4}.$$

Insert next

$$1 = \prod_{n=1}^N (\chi_n + (1 - \chi_n)) := \prod_{n=1}^N (\chi_n + \chi_n^c)$$

in the expectation in (87) and expand to get

$$P_N = \sum_{A \subset \{1, \dots, N\}} \mathbb{E} \chi_{D_N} \chi_A \chi_{A^c}^c$$

where  $\chi_A = \prod_{n \in A} \chi_n$  and  $\chi_{A^c}^c = \prod_{n \in A^c} \chi_n^c$ . On the support of  $\chi_{D_N} \chi_A \chi_{A^c}^c$  one has

$$N\delta \geq \sum_n m_n s_n \sigma_n \geq \delta^{\frac{1}{2}} \sum_{n \in A} m_n$$

so

$$(89) \quad P_N \leq \sum_{|A| > \alpha N} \mathbb{E} \chi \left( \sum_{n \in A} m_n \leq \delta^{\frac{1}{2}} N \right) + \sum_{|A| \leq \alpha N} \mathbb{E} \chi_{A^c}^c,$$

where we choose  $\alpha := \min(1, a)/8$  with  $a$  taken from Proposition 4.3 (ii). Observe that  $\alpha$  is independent of  $\rho, \delta$  and  $N$ .

Let us consider the two sums on the RHS of (89) in turn. For the first one we use independence: let  $m_A := \sum_{n \in A} m_n$  then

$$(90) \quad P(m_A < \delta^{\frac{1}{2}} N) \leq e^{\delta^{\frac{1}{2}} t N} \mathbb{E} e^{-t m_A} = e^{\delta^{\frac{1}{2}} t N} \prod_{n \in A} \mathbb{E} e^{-t m_n}.$$

By Proposition 4.3 (i)

$$(91) \quad \mathbb{E} e^{-t m_n} \leq Cx + e^{-tx} \leq 2e^{-tx(t)}$$

where the auxiliary variable  $x = x(t)$  is chosen so that  $Cx(t) = e^{-tx(t)}$ . Here  $x(t) \rightarrow 0$  and  $tx(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Thus assuming  $\delta$  small enough and taking  $t = t(\delta)$  such that  $x(t) = 2\delta^{\frac{1}{2}}/\alpha$ , in the case  $|A| \geq \alpha N$  the right side of (90) is bounded by  $2^N e^{-\delta^{\frac{1}{2}} t(\delta) N}$  where  $\delta^{\frac{1}{2}} t(\delta) \rightarrow \infty$  as  $\delta \rightarrow 0$ . Hence

$$(92) \quad \sum_{|A| > \alpha N} \mathbb{E} \chi \left( \sum_{n \in A} m_n \leq \delta^{\frac{1}{2}} N \right) \leq 2^N e^{-g(\delta) N}.$$

where  $g(\delta) \rightarrow \infty$  as  $\delta \rightarrow 0$ .

For the second sum in (89) we need to bound

$$\mathbb{E} \chi_B^c := \mathbb{E} \prod_{n \in B} (1 - \chi_n)$$

for  $|B| \geq (1 - \alpha)N$ . For that purpose, we shall make use of the elementary identity

$$(93) \quad 1 - \prod_{j=1}^{\infty} (1 - a_j) = \sum_{j=1}^{\infty} a_j \prod_{r=1}^{j-1} (1 - a_r),$$

valid for any sequence  $(a_j)_{j \geq 1}$  with  $a_j \in [0, 1]$  for all  $j \geq 1$ . Recall eq. (88) and denote  $\chi_{n,m}^c := 1 - \chi_{n,m}$ . We also set  $\chi_{n,m}^c := 0$  for  $m < 0$ . For any fixed  $n$  arrange the variables  $\chi_{n,m}^c$  with  $m \in \mathbb{Z}$  into a sequence in some order and apply the identity (93) to write

$$(94) \quad 1 - \chi_n = 1 - \prod_{m \in \mathbb{Z}, m \neq n} (1 - \chi_{n,m}^c) = \sum_{\ell \in \mathbb{Z}, \ell \neq 0} \chi_{n,n+\ell}^c \tilde{\chi}_{n,\ell},$$

with certain variables  $\tilde{\chi}_{n,\ell}$  satisfying  $0 \leq \tilde{\chi}_{n,\ell} \leq 1$ . Let us denote

$$(95) \quad \chi_n^+ := \sum_{\ell > 0} \chi_{n,n+\ell}^c \tilde{\chi}_{n,\ell} \quad \text{and} \quad \chi_n^- := \sum_{\ell < 0} \chi_{n,n+\ell}^c \tilde{\chi}_{n,\ell}.$$

Then  $\chi_n^\pm \leq 1$  (since  $\chi_n^+ + \chi_n^- = 1 - \chi_n$ ) and

$$(96) \quad \chi_n^\pm \leq \sum_{\pm \ell > 0} \chi_{n,n+\ell}^c.$$

We may then estimate

$$(97) \quad \begin{aligned} \prod_{n \in B} (1 - \chi_n) &= \prod_{n \in B} (\chi_n^+ + \chi_n^-) = \sum_{(s_n = \pm)_{n \in B}} \prod_{n \in B} \chi_n^{s_n} \\ &\leq \sum_{s: N_+ > (1-2\alpha)N} \prod_{n: s_n = +} \chi_n^+ + \sum_{s: N_+ \leq (1-2\alpha)N} \prod_{n: s_n = -} \chi_n^- \end{aligned}$$

where  $N_+$  is the number of  $n$  in the set  $B$  such that  $s_n = +$ . We estimate the expectations of the two products on the RHS in turn.

For the first product, let  $D \subset \{1, \dots, N\}$  with  $p := |D| \geq (1 - 2\alpha)N$ . List the elements of  $D$  as  $n_1 < n_2 < \dots < n_p$ . Then, as  $0 \leq \chi_{n_j}^+ \leq 1$ ,

$$(98) \quad \mathbb{E} \chi_{n_1}^+ \cdots \chi_{n_p}^+ \leq \sum_{\ell_1 > 0} \mathbb{E} \chi_{n_1, n_1 + \ell_1}^c \chi_{n_2}^+ \cdots \chi_{n_p}^+ \leq \sum_{\ell_1 > 0} \mathbb{E} \chi_{n_1, n_1 + \ell_1}^c \chi_{n_{i_2}}^+ \cdots \chi_{n_p}^+,$$

where  $n_{i_2}$  is the smallest  $n_j$  larger than  $n_1 + \ell_1$ . Iterating we get

$$(99) \quad \mathbb{E} \chi_{n_1}^+ \cdots \chi_{n_p}^+ \leq \sum_{r=1}^p \sum_{(\ell_1, \dots, \ell_r)} \mathbb{E} \prod_{j=1}^r \chi_{n_{i_j}, n_{i_j} + \ell_j}^c$$

where  $n_{i_{j+1}}$  is the smallest  $n_j$  larger than  $n_{i_j} + \ell_j$  and  $n_{i_1} = n_1$ . As the intervals  $[n_j, n_j + \ell_j]$  cover the set  $D$ , the  $r$ -tuples  $(\ell_1, \dots, \ell_r)$  in the above sum satisfy

$$(100) \quad \sum_{j=1}^r \ell_j \geq p - r$$

Next, by Proposition 4.3 (ii) the factors in the product in (99) are independent and thus

$$\mathbb{E} \prod_{j=1}^r \chi_{n_{i_j}, n_{i_j} + \ell_j}^c = \prod_{j=1}^r \mathbb{E} \chi_{n_{i_j}, n_{i_j} + \ell_j}^c$$

From (86) and (88) we deduce

$$\mathbb{E} \chi_{n_j, n_j + \ell_j}^c \leq C(\rho) \delta^{q/4} (2^q \rho^{1+a})^{\ell_j}$$

whereby

$$(101) \quad \mathbb{E} \chi_{n_1}^+ \cdots \chi_{n_p}^+ \leq \sum_{r=1}^p \sum_{(\ell_1, \dots, \ell_r)} (C(\rho) \delta^{q/4})^r (2^q \rho^{1+a})^{\sum \ell_j}$$

Using (100) we see that RHS is bounded by

$$\rho^{(1+a/2)p} \sum_{r=1}^p (C(\rho) \delta^{q/4})^r \sum_{(\ell_1, \dots, \ell_r)} (2^q \rho^{a/2})^{\sum \ell_j}$$

For an upper bound drop the constraints on  $\ell_i$  to bound (101) by

$$\rho^{(1+a/2)p} \sum_{r=1}^p (C(\rho) \delta^{q/4})^r \left( \sum_{\ell=1}^{\infty} (2^q \rho^{a/2})^{\ell} \right)^r$$

Choosing first  $\rho$  small enough and then  $\delta \leq \delta(\rho)$  this is bounded by

$$C(\rho) \delta^{1/4} \rho^{(1+a/2)p} \leq C(\rho) \delta^{1/4} \rho^{(1+a/2)(1-2\alpha)N} \leq C(\rho) \delta^{1/4} \rho^{(1+2b)N}$$

for a constant  $b > 0$  by our choice of  $\alpha$ . The expectation of the first sum in eq. (97) is then bounded by

$$(102) \quad C(\rho) 2^N \delta^{1/4} \rho^{(1+2b)N}.$$

Consider finally the second sum in eq. (97). We proceed as for the first sum this time considering a set  $D \subset \{1, \dots, N\}$  with elements  $n_1 > n_2 > \dots > n_p$  with  $p \geq \alpha N$ . Now we write  $\chi_{n_1}^- \leq \sum_{\ell_1 > 0} \chi_{n_1, n_1 - \ell_1}^c$  and end up with the analogue of eq. (99):

$$(103) \quad \mathbb{E} \chi_{n_1}^- \cdots \chi_{n_p}^- \leq \sum_{r=1}^p \sum_{(\ell_1, \dots, \ell_r)} \prod_{j=1}^r \mathbb{E} \chi_{n_{i_j}, n_{i_j} - \ell_j}^c$$

where  $n_{i_{j+1}}$  is the largest  $n_j$  smaller than  $n_{i_j} - \ell_j$  and  $n_{i_1} = n_1$ , and this time Proposition 4.2 was used for independence. From the same Proposition we also get

$$\mathbb{E} \chi_{n, n-\ell}^c \leq c e^{-c 2^{-2\ell} \rho^{-\ell + \frac{1}{2}} (\log \delta)^2}.$$

For small enough  $\rho$  we have  $2^{-2\ell} \rho^{-\ell + \frac{1}{2}} \geq (\ell + \rho^{-\frac{1}{8}}) \rho^{-\frac{1}{8}}$  for all  $\ell \geq 1$ . Hence

$$\prod_{j=1}^r \mathbb{E} \chi_{n_{i_j}, n_{i_j} - \ell_j}^c \leq c^r \exp\left(-c(\log \delta)^2 \rho^{-\frac{1}{8}} \left(r \rho^{-\frac{1}{8}} + \sum_{j=1}^r \ell_j\right)\right)$$

As  $\rho < 1$ , by (100) we also have

$$r \rho^{-\frac{1}{8}} + \sum_{j=1}^r \ell_j \geq (p + \sum_{j=1}^r \ell_j)/2.$$

Thus

$$\prod_{j=1}^r \mathbb{E} \chi_{n_{i_j}, n_{i_j} - \ell_j}^c \leq \exp(c \rho^{-\frac{1}{8}} (\log \delta)^2 p/2) c^r \exp(-c(\log \delta)^2 \rho^{-\frac{1}{8}} \sum_{j=1}^r \ell_j)$$

Now recall that  $p \geq \alpha N$ , take  $\delta$  small enough, and proceed as above by summing first over the  $\ell_j$ 's, and then performing a geometric sum over  $r$  in order to conclude that the second sum in (97) has the upper bound

$$2 \exp(-c\rho^{-\frac{1}{8}}(\log \delta)^2 N\alpha/2)$$

For small  $\delta$  this is by far dominated by the bound (102), and therefore

$$(104) \quad \mathbb{E} \chi_B^c \leq 2^{N+1} \delta^{1/4} \rho^{(1+2b)N}.$$

Going back to equation (89), and recalling (92) with (102) and (104), we conclude that for  $\delta \leq \delta(\rho)$

$$(105) \quad P_N \leq 2^{2N+2} \delta^{1/4} \rho^{(1+2b)N}.$$

which gives the claim of Theorem 4.1.  $\square$

**4.4. Proofs of the Propositions.** We will now prove the Propositions 4.2 and 4.3 of Subsection 4.2 describing the statistics of  $m_n$ ,  $L_{n,m}$  and  $t_{n,k}$ . We start by noting that the random measures  $\nu_n(\cdot)$  and  $\rho^{n-1}\nu_1(\rho^{1-n}\cdot)$  are equal in law. Especially, the  $m_n$  are i.i.d. and it suffices to study  $m_1$ . Similarly  $\zeta_k|_{B_n}$  equals in law with  $\zeta_1|_{B_{n-k+1}}$  and thus  $t_{n,k}$  equals  $t_{n-k+1,1}$  in law. The value  $k = 0$  is slightly different, but it can be treated exactly in the same manner as the case  $k \geq 1$ . Finally,  $L_{n,m}$  and  $L_{1,m-n+1}$  are equal in law.

We need first the following Lemma.

**Lemma 4.4.** *There exists  $q, q_1 > 1$  and  $C > 0$  (each independent of  $\rho$ ) such that for all intervals  $J, I \subset [-1/4, 1/4]$  satisfying  $|J| \leq 2|I|$ , and with mutual distance at most  $100|I|$ , one has*

$$(106) \quad \mathbb{P}(\delta_\nu(J, I) > \lambda) \leq C\lambda^{-q} \left( \frac{|J|}{|I|} \right)^{q_1}.$$

**Proof.** We use the comparison (52) with the measure  $\eta$  in order to estimate

$$(107) \quad \nu(J)/\nu(I) \leq G_2 \eta(J)/\eta(I),$$

where we recall that all the moments of the variable  $G_2$  are finite. Next, in case  $|I| \leq 1/100$  we may scale further by using the exact scaling law (56), and apply the translation invariance of  $\eta$  to deduce that  $\eta(J)/\eta(I) \sim \eta(J')/\eta(I')$ , where now  $I', J' \subset [0, 1]$  with  $1/100 \leq |I'|$  and  $|J'| \leq |J|/|I| \leq 100|J'|$ . In the case  $|I| \geq 1/100$  no scaling is needed.

In this situation, if  $r < \infty$  it follows from Proposition 6.1 that  $\eta(I')^{-1} \in L^r$  uniformly with respect to the  $I'$ . We can thus fix exponents  $1 < q < \tilde{q} < p < 2/\beta^2$  and get by (107), Hölder's inequality and Theorem 3.7

$$(108) \quad \begin{aligned} \|\nu(J)/\nu(I)\|_q &\leq C\|\eta(J)/\eta(I)\|_{\tilde{q}} = C\|\eta(J')/\eta(I')\|_{\tilde{q}} \\ &\leq C\|\eta(J')\|_p \leq C(|J|/|I|)^{\zeta(p)/p}, \end{aligned}$$

where  $\zeta(p) > 1$ . The constant  $C$  depends only on the exponents  $q, \tilde{q}$  and  $p$ . Thus

$$(109) \quad \mathbb{P}(\delta_\nu(J, I) > \lambda) \leq C\lambda^{-q}(|J|/|I|)^{q\zeta(p)/p}$$

The desired bound follows by choosing the exponent  $q > 1$  close enough to  $p$  in order to ensure that  $q_1 := q\zeta(p)/p > 1$ .  $\square$

Let us then discuss  $m_1$ . Observe that the denominator of the integrand in (82) can be dominated as follows:

$$(110) \quad 1 + L_{1,1} + L_1(r) \leq 1 + L_{1,1} + \sum_{m=0}^{\infty} 2^{-m} k_m(r)$$

where for  $r \in (\rho, 2\rho)$  and  $m \geq 0$  one sets

$$(111) \quad k_m(r) := \sum_{I \in \mathcal{D}_{p+m}} K_{1,I} 1_{C_I \cap S_r \neq \emptyset}.$$

For any fixed  $r \in (\rho, 2\rho)$  the sum (111) has at most four non-zero terms.

For  $m \geq 0$  denote by  $\mathcal{H}_m$  the set of all pairs  $\mathbf{J} = (J_1, J_2)$  that contribute to  $k_m(r)$  in (111) for some  $r \in (\rho, 2\rho)$ . To estimate  $\delta_{\nu_1}(\mathbf{J})$ , we may scale by the factor  $\rho^{-\frac{1}{2}}$  in order to consider instead the identically distributed quantity  $\nu(J'_1)/\nu(J'_2)$ , where now  $J'_1, J'_2 \subset [-1/4, 1/4]$ . Thus Lemma 4.4 applies. As we additionally have  $|J_1| = |J_2|$ , there is  $q > 1$  and a constant  $C > 0$  such that

$$(112) \quad \mathbb{P}(\delta_{\nu_1}(\mathbf{J}) > R) \leq CR^{-q} \quad \text{for all } \mathbf{J} \in \cup_{m \geq 0} \mathcal{H}_m.$$

Choose next  $\alpha > 0$  and  $\gamma \in (0, 1)$  such that  $4\alpha \sum_m 2^{m(\gamma-1)} \leq 1$  together with  $\gamma q > 1$ . Fix  $R > 0$ . We observe that by these choices

$$\delta_{\nu_1}(\mathbf{J}) \leq \alpha 2^{\gamma m} R \quad \text{for all } \mathbf{J} \in \mathcal{H}_m, m \geq 0 \implies L_1(r) \leq R \quad \text{for all } r \in (\rho, 2\rho).$$

Since we have the obvious estimate  $\#(\mathcal{H}_m) \leq c2^m$  for the number of the pairs in  $\mathcal{H}_m$ , by combining the above implication with the uniform estimate (112) one may estimate

$$\begin{aligned} \mathbb{P}(L_1(r(\sigma)) > R \text{ for some } r \in (\rho, 2\rho)) &\leq \sum_{m=0}^{\infty} \sum_{\mathbf{J} \in \mathcal{H}_m} \mathbb{P}(\delta_{\nu_1}(\mathbf{J}) > \alpha 2^{\gamma m} R) \\ &\leq CR^{-q} \sum_m c2^m 2^{-q\gamma m} \leq CR^{-q}. \end{aligned}$$

In a similar vain we may apply Lemma 4.4 to immediately obtain the corresponding tail estimate for  $L_{1,1}$ . Indeed, by (80) this depends only on a finite ( $\rho$ -independent) number of ratios  $\delta_{\nu_1}(I_1, I_2)$ , with  $I_1, I_2 \subset [-4\rho, 4\rho]$  and  $|I_1|, |I_2| \geq 2^{-7}\rho$ , see (79). Putting things together, we obtain (for  $R > 1$ , say) the bound

$$(113) \quad \mathbb{P}(m_1 < 1/R) \leq CR^{-q} \leq CR^{-1},$$

with  $C$  is independent of  $\rho$ .

Consider next  $L_{n,m}$  with  $m > n$  and use  $L_{n,m} \sim L_{1,m-n+1}$ . By (82)  $L_{1,m-n+1}$  is bounded from above by a sum of terms (with  $\rho$ -independent upper bound for their number)

$$\nu_1(J)/\nu_1(I)$$

where  $2^{-8}\rho \leq |I| \leq 2^{-4}\rho$  and  $|J| \leq \rho^{m-n+\frac{1}{2}}$ , and in addition  $I, J \subset [-4\rho, 4\rho]$ . The constant  $C$  above is independent of  $m, n$  and  $\rho$ . Via scaling the desired bound (86) is now a direct consequence of Lemma 4.4, as we observe that  $|J|/|I| \leq C\rho^{m-n-\frac{1}{2}}$ .

Finally we turn to  $t_{n,1}$  given in (72). By scaling we may take the sup and the inf over  $x \in B_n \cap \mathbb{R}$  of  $\exp(\beta\tilde{\psi})$  where  $\tilde{\psi} := \psi(\cdot, \rho^{3/2}, \rho^{1/2})$  and we may replace there  $\tilde{\psi}$

by  $\widehat{\psi} := \widetilde{\psi}(\cdot) - \widetilde{\psi}(0)$ . The covariance of  $\widehat{\psi}$  is clearly  $c\rho^{-3/2}$ -Lipschitz and length of the interval  $B_n \cap \mathbb{R}$  is  $8\rho^n$ . Lemma 3.3 yields that

$$\mathbb{P}(|\widetilde{\psi}| > \lambda c \rho^{n/2-3/4}) \leq C(1 + \lambda)e^{-\frac{1}{2}\lambda^2},$$

which finishes the proof of the remaining Proposition 4.2.  $\square$

**4.5. Integrability of  $K_\nu$ .** In next section we shall also make use of the the following observation:

**Lemma 4.5.** *Let  $\beta < \sqrt{2}$ . Then almost surely  $K_\nu \in L^1([0, 1] \times [0, 2])$ .*

**Proof.** Recall that  $S = \mathbb{R} \times [0, 2]$  is tiled by the Whitney squares  $C_I$ . By definition, on such a square  $K_\nu$  is a finite sum of ratios  $\nu(J_1)/\nu(J_2)$  with  $|J_1| = |J_2| \leq 2^{-4}$  and of controlled mutual distance as in Lemma 4.4. Thus, for  $|J_i|$  small enough  $J_i$  lie on a common interval of length  $\frac{1}{2}$  and we have a uniform bound for  $\mathbb{E} \nu(J_1)/\nu(J_2) \leq \|\nu(J_1)/\nu(J_2)\|_q$ ,  $q < 2/\beta^2$  from Lemma 4.4 (or more directly from (108)) and for the finitely many ones not fitting to such interval we use again (108). Hence there is also a uniform bound for  $\mathbb{E} K_\nu(I)$  and one obtains

$$\mathbb{E} \int_{[0,1] \times [0,2]} K_\nu \leq \sum_{I \subset D([0,1])} |C_I| \mathbb{E} K_\nu(I) \leq C \sum_I |C_I| < \infty. \quad \square$$

## 5. CONCLUSION OF THE PROOF

In this final section we give a precise formulation to our main result as a Theorem and prove it using the work done in the previous sections. In order to make the setup clear, let us recall that our random circle homeomorphism was defined in Section 3 via formulae (57) and (58). Its extension to the unit disc is constructed by the method described in Section 2.4, and formula (24) in particular.

The welding method described in Section 2 requires estimates for the Lehto integral of the distortion function in  $\mathbb{D}$ . Theorem 2.6 reduces these bounds to the boundary function, and here the crucial estimates are provided by our Theorem 4.1 in Section 4.

**Theorem 5.1.** *Let  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  be the random circle homeomorphism from Definition 3.8, and let  $\Psi : \mathbb{D} \rightarrow \mathbb{D}$  be its extension as in (20) - (24). Let  $\mu = \mu_\Psi := \partial_{\bar{z}}\Psi/\partial_z\Psi$  be the complex dilatation of the extension on  $\mathbb{D}$ , and set  $\mu = 0$  outside  $\mathbb{D}$ .*

*Then almost surely there exists a (random) homeomorphic  $W_{loc}^{1,1}$ -solution  $f : \mathbb{C} \rightarrow \mathbb{C}$  to the Beltrami equation*

$$(114) \quad \partial_{\bar{z}}f = \mu\partial_zf \quad \text{a.e. in } \mathbb{C},$$

*that satisfies the normalization  $f(z) = z + o(1)$  as  $z \rightarrow \infty$ . Moreover, there exist  $\alpha > 0$  such that the restriction  $f : \mathbb{T} \rightarrow \mathbb{C}$  is a.s.  $\alpha$ -Hölder continuous.*

**Proof.** We sketch the proof along the lines of [4, Thm 20.9.4], to which presentation we refer for further details and background.

For any integer  $n \geq 1$  choose  $N_n = \lceil \rho^{-(1+\frac{1}{2}b)n} \rceil \in \mathbb{N}$  where  $b$  is as in Theorem 4.1. Denote

$$\zeta_{n,k} := \exp(2\pi i k / N_n) \quad \text{for } k = 1, \dots, N_n.$$

Write also  $G_n := \{\zeta_{n,1}, \dots, \zeta_{n,N_n}\}$ . Thus the distance on  $\mathbb{T}$  to the set  $G_n$  is bounded by  $2\pi/N_n \sim \rho^{(1+\frac{1}{2}b)n}$ , up to a constant.

For a given  $n \geq 1$  and  $k \in 1, \dots, N_n$  let us denote by  $A_{n,k}$  the event

$$A_{n,k} = \{\omega : L_{K_\nu}(k/N_n, \rho^n, 2\rho) < n\delta\},$$

and set  $A_n = \bigcup_{k=1}^{N_n} A_{n,k}$ . Note that here we consider Lehto integrals in the half plane. Theorem 4.1(i) combined with stationarity yields that

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{N_n} \mathbb{P}(A_{n,k}) \leq \sum_{n=1}^{\infty} N_n c(\delta) \rho^{(1+b)n} \leq c(\delta) \sum_{n=1}^{\infty} \rho^{\frac{b}{2}n} < \infty.$$

Borell-Cantelli lemma yields that almost every  $\omega$  belongs to the complement of the event  $\bigcup_{n > n_0(\omega)} A_n$ .

Also, we obtain by Lemma 3.6 that

$$K_\tau \leq E^2 K_\nu,$$

where almost surely  $E < \infty$ . From Theorem 2.6 and (59) we see that  $K(z, F)$ , the distortion of the extension of  $h$ , is bounded by a constant times  $K_\tau(z)$ . Hence Lemma 4.5 implies that almost surely

$$\int_{[0,1] \times [0,2]} K(z, F) \leq C_0 \int_{[0,1] \times [0,2]} K_\tau \leq C_0 E^2 \int_{[0,1] \times [0,2]} K_\nu < \infty.$$

We may thus forget the probabilistic setup by fixing an event  $\omega_0 \in \Omega$  so that we are in the following situation: We are given the complex dilatation  $\mu$  on  $\mathbb{D}$ , so that the distortion  $K = (1 + |\mu|)/(1 - |\mu|)$  satisfies pointwise

$$K(e^{2\pi iz}) \leq C_0 K_\tau(z) \leq C_0 E(\omega_0)^2 K_\nu(z), \quad z \in \mathbb{H}.$$

Further, from the definition in (24) we have  $K \equiv 1$  for  $|z| \leq e^{-4\pi}$ . We also have  $K_\nu \in L^1 \cap L_{loc}^\infty$  on the square  $[0, 1] \times [0, 2]$ , and for each  $n \geq n_0$  and  $k \in 1, \dots, N_n$  it holds that

$$L_{K_\tau}(k/N_n, \rho^n, 2\rho) \geq (E(\omega_0))^{-2} L_{K_\nu}(k/N_n, \rho^n, 2\rho) \geq n\delta(E(\omega_0))^{-2} =: n\delta'.$$

We next proceed as in the standard proof of Lehto's theorem by approximating  $\mu$  by e.g. the sequence  $\mu_\ell := \frac{\ell}{\ell+1} \mu$ ,  $\ell \in \mathbb{N}$ . Letting  $f_\ell$  denote the corresponding normalized solution of the Beltrami equation with coefficient  $\mu_\ell$ , i.e. with the asymptotics  $f_\ell(z) = z + o(1)$  as  $z \rightarrow \infty$ , then every  $f_\ell$  is a quasiconformal homeomorphism of  $\mathbb{C}$ .

To show that (114) has a homeomorphic  $W^{1,1}$ -solution, we need to control the approximations  $f_\ell$ . For this we first apply [4, Lemma 20.2.3], which tells that the inverse maps  $g_\ell = f_\ell^{-1}$  have the following modulus of continuity,

$$|g_\ell(z) - g_\ell(w)| \leq 16\pi^2 \frac{|z|^2 + |w|^2 + \int_{\mathbb{D}} \frac{1+|\mu_\ell(\zeta)|}{1-|\mu_\ell(\zeta)|} d\zeta}{\log(e + \frac{1}{|z-w|})}, \quad z, w \in \mathbb{C}.$$

Here the integrals are uniformly bounded as

$$\frac{1 + |\mu_\ell(\zeta)|}{1 - |\mu_\ell(\zeta)|} \leq K(\zeta) \leq C_0 K_\tau(z), \quad \zeta = e^{2\pi iz},$$



and  $K_\tau \in L^1[0, 1] \times [0, 2]$ . Thus the inverse maps  $g_\ell = f_\ell^{-1}$  form an equicontinuous family.

In order to check the equicontinuity of the family  $(f_\ell)_{\ell \geq 1}$  itself we first consider a point  $z \in \mathbb{D}$ . Writing  $2a = 1 - |z|$ , observe that  $K$  is bounded in  $B(z, a)$  and as  $K_\ell := K(\cdot, f_\ell) \leq K$  we have for any  $\ell \geq 1$  and  $u \in (0, a/2)$

$$\begin{aligned} L_{K_\ell}(z, u, 1) &\geq L_K(z, u, a) \geq (\|K\|_{L^\infty(B(z, a))})^{-1} \log(a/u) \\ &\rightarrow \infty \quad \text{as } u \rightarrow 0. \end{aligned}$$

Moreover, by Koebe's theorem or [4, Cor. 2.10.2] we obtain

$$(115) \quad f(2\mathbb{D}) \subset 5\mathbb{D}.$$

Thus  $\text{diam}(f_\ell(B(z, 1))) \leq 5$ , which may be combined with Lemma 2.3 to obtain

$$\text{diam}(f_\ell(B(z, u))) \rightarrow 0 \quad \text{as } u \rightarrow 0, \quad \text{uniformly in } \ell.$$

This proves the equicontinuity at interior points  $z \in \mathbb{D}$ . Equicontinuity at exterior points follows e.g. from Koebe's theorem.

In order to next consider the uniform behaviour on  $\mathbb{T}$ , note that it suffices to prove local equicontinuity on points of  $[0, 1]$  for the family

$$F_\ell(z) = f_\ell(e^{2\pi iz}), \quad \ell \in \mathbb{N}.$$

We first estimate the diameter of the image  $F_\ell(B(k/N_n, \rho^n))$ , assuming that  $n \geq n_0$ . Applying the fact  $\text{diam } F_\ell(B(k/N_n, 2\rho)) \leq \text{diam}(f_\ell(B(\zeta_{n,k}, 1))) \leq 5$  and using this together with Lemma 2.3 we obtain

$$(116) \quad \begin{aligned} \text{diam}(F_\ell(B(k/N_n, \rho^n))) &\leq \text{diam}(F_\ell(B(k/N_n, 2\rho))) 16 \exp(-2\pi^2 n \delta') \\ &\leq 80e^{-nc'}. \end{aligned}$$

From these estimates we get the required equicontinuity. Namely, working now on the circle  $\mathbb{T}$ , since the set  $G_n$  is evenly spread on  $\mathbb{T}$ , the balls  $B(\zeta_{n,k}, \rho^{n+1})$  cover a  $\rho^{n+2}$ -neighbourhood of  $\mathbb{T}$  in such a way that any two points that are in this neighbourhood, with distance not exceeding  $\rho^{n+2}$ , lie in the same ball. Since this holds for every  $n \geq n_0$  we infer from (116) that there is  $\varepsilon_0 > 0$  and  $\alpha > 0$  so that, uniformly in  $\ell$

$$(117) \quad |f_\ell(z) - f_\ell(w)| \leq C|z - w|^\alpha \quad \text{if } |z| = 1, \quad \text{and} \quad \begin{cases} 1 - \varepsilon_0 \leq |w| \leq 1 + \varepsilon_0 \\ |z - w| \leq \varepsilon_0. \end{cases}$$

One may actually take  $\alpha = c'/\log(1/\rho)$ . This clearly yields equicontinuity at the points of  $\mathbb{T}$ , and hence on  $\widehat{\mathbb{C}}$ . We may now pass to a limit and one obtains  $W^{1,1}$ -homeomorphic solution  $f(z) = \lim_{\ell \rightarrow \infty} f_\ell(z)$  to the Beltrami equation as in [4, p. 585].

At the same time the estimate (117) shows that  $f : \mathbb{T} \rightarrow \mathbb{C}$  is Hölder continuous. Since  $f$  is analytic outside the disk, with  $f(z) = z + o(1)$  at infinity, in fact it follows that  $f$  is Hölder continuous on  $\mathbb{C} \setminus \mathbb{D}$ .  $\square$

Collecting the results established we now arrive at the main theorem of this paper.

**Theorem 5.2.** *Let  $\phi = \phi_\omega$  be the random circle homeomorphism, with derivative the exponentiated GFF, as defined in (57) and (58).*

*Then for  $\beta^2 < 2$  and almost surely in  $\omega$ , the mapping  $\phi$  admits a conformal welding. That is, there are a random Jordan curve*

$$(118) \quad \gamma = \gamma_{\omega, \beta}$$

and conformal mappings  $f_{\pm}$  onto the complementary domains of  $\gamma$ , such that  $\phi = f_+^{-1} \circ f_-$  on  $\mathbb{T}$ .

Moreover, almost surely in  $\omega$ , the Jordan curve  $\gamma$  in (118) is unique, up to composing with a Möbius transformation  $\Gamma = \Gamma_{\omega}$  of the Riemann sphere.

**Proof.** We argue as in Section 2. Using the complex dilatation of the extension  $\Psi$  from Theorem 5.1, we find a homeomorphic solution  $f$  to the auxiliary equation (114). This is conformal outside the disk, so we set  $f_- = f|_{\mathbb{C} \setminus \mathbb{D}}$ . Inside the disk  $K(z, f)$  is locally bounded, so the uniqueness of the Beltrami equation gives  $f(z) = f_+ \circ \Psi(z)$ ,  $z \in \mathbb{D}$ , where  $f_+$  is a conformal homeomorphism on  $\mathbb{D}$ . Since the boundary  $\partial f_+(\mathbb{D}) = \partial f_-(\mathbb{C} \setminus \mathbb{D}) = f(\mathbb{T}) = \gamma$  is a Jordan curve,  $f_{\pm}$  extend to  $\mathbb{T}$  where we have

$$\phi = (f_+)^{-1} \circ f_-.$$

Finally, according to the proof of Theorem 5.1  $f_-$  is Hölder continuous in  $\mathbb{C} \setminus \mathbb{D}$ , thus the uniqueness of the welding curve follows from the Jones-Smirnov Theorem 2.4.  $\square$

## 6. APPENDIX: NEGATIVE MOMENTS

Here we prove the finiteness of all negative moments for the measure  $\eta$ , that was defined in the proof of Theorem 3.7.

**Proposition 6.1.** Suppose  $\beta^2 < 2$ . Then

$$\mathbb{E}(\eta(I)^{-q}) \leq C < \infty, \quad 0 < q < \infty,$$

for a constant  $C = C(q, |I|)$  depending only on the exponent  $q$  and the length  $|I|$ .

**Proof.** Fix  $t > 0$ . Define for  $\varepsilon > 0$  the set  $U_{\varepsilon, t}$  by setting  $U_{\varepsilon, t} := U \cap \{\varepsilon < y \leq t\}$ . As in (53) one deduces the existence of the limit measure

$$(119) \quad \eta_t(dx) := \lim_{\varepsilon \rightarrow 0^+} \exp(\beta U_{\varepsilon, t}(x) - (\beta^2/2)\text{Cov}(U_{\varepsilon, t}))dx.$$

Denote  $M := \eta_{1/2}([0, 1])$ ,  $M_1 := \eta_{1/8}([0, 1/4])$  and  $M_2 := \eta_{1/8}([3/4, 1])$ . By scaling and translation invariance the random variables  $M_1, M_2$  and  $M$  are identically distributed. Moreover, by comparing the exponents as in the proof of Lemma 3.5, we see that

$$(120) \quad M \geq B(M_1 + M_2),$$

where  $B := \exp(\inf_{x \in [0, 1]} \beta U_{1/8, 1/2}(x) - (\beta^2/2)\text{Cov}(U_{1/8, 1/2}))$  has all moments finite. By construction, the random variables  $M_1, M_2$  and  $B$  are independent.

Similarly, by comparing  $\eta$  and  $\eta_{1/2}$  we see that it is enough to prove

$$(121) \quad \mathbb{E} M^{-q} < \infty \quad \text{for } q > 0.$$

We first prove this for small values of  $q$ . For that end, consider for  $s > 0$  the Laplace transform

$$(122) \quad \begin{aligned} \Psi_M(s) &:= \mathbb{E} \exp(-sM) \leq \mathbb{E}(-sB(M_1 + M_2)) \\ &\leq \mathbb{E} \Psi_{M_1}(sB) \Psi_{M_2}(sB) = \mathbb{E}(\Psi_M(sB))^2. \end{aligned}$$

Since especially  $\mathbb{E} B^{-1} < \infty$ , we may estimate  $\mathbb{P}(B < 1/s) \leq c/s$ . By substituting  $s^2$  in place of  $s$  in (122) and applying this inequality we obtain

$$(123) \quad \Psi_M(s^2) \leq c/s + \Psi_M^2(s),$$

where one may assume that  $c \geq 2$ .

Denote  $f(s) := (c/s^{1/2} + \Psi_M(s))$ . Then (123) yields

$$(124) \quad f(s^2) = c/s + \Psi_M(s^2) \leq f^2(s).$$

Since  $\Psi_M(s) \rightarrow 0$  as  $s \rightarrow \infty$  (while  $\mathbb{P}(M = 0) = 0$ ), we may choose  $s_0 > 0$  with  $\Psi_M(s_0) \leq 1/2$ , whence (124) iterates to  $f(s_0^{2^k}) \leq 2^{-2^k}$  for  $k \geq 1$ . Together with monotonicity of  $f$  this yields  $\delta > 0$  such that  $f(s) \leq cs^{-\delta}$  for  $s > 0$ , especially  $\Psi_M(s) \leq cs^{-\delta}$ .

We obtain that

$$\mathbb{E} M^{-\delta/2} = c \int_0^\infty \mathbb{E} e^{-sM} s^{\delta/2-1} ds < \infty.$$

In order to cover all values of  $q$  in (121) we employ a simple bootstrapping argument. Assume that  $\mathbb{E} M^{-q} < \infty$  for some  $q > 0$ . By applying the inequality between the arithmetic and geometric mean, the independence of  $B, M_1$  and  $M_2$ , and the fact that  $B$  has all negative moments finite, we may estimate

$$(125) \quad \mathbb{E} M^{-2q} \leq \mathbb{E} (B(M_1 + M_2))^{-2q} \leq c \mathbb{E} (M_1 M_2)^{-q} = c(\mathbb{E} (M)^{-q})^2 < \infty.$$

By induction, this finishes the proof.  $\square$

## REFERENCES

- [1] H. Airault, P. Malliavin and A. Thalmaier: *Canonical Brownian motion on the space of univalent functions and resolution of Beltrami equations by a continuity method along stochastic flows*. J. Math. Pures Appl. 83 (2004), 955–1018.
- [2] R.J. Adler and J.E. Taylor: *Random fields and geometry*. Springer, New York, 2007.
- [3] G. Anderson, M. Vamanamurthy and M. Vuorinen: *Conformal invariants, inequalities, and quasiconformal maps*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., 1997.
- [4] K. Astala, T. Iwaniec and G. Martin: *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*. Princeton Mathematical Series 47, Princeton University Press, 2009.
- [5] E. Bacry and J. F. Muzy: *Log-infinitely divisible multifractal processes*, Comm. Math. Phys. 236 (2003), 449–475.
- [6] A. Beurling and L.V. Ahlfors: *The boundary correspondence under quasiconformal mappings*. Acta Math. 96 (1956), 125–142.
- [7] I. Benjamini and O. Schramm: *KPZ in one dimensional random geometry of multiplicative cascades*, Comm. Math. Phys. 289 (2009), 653–662.
- [8] J. Cardy: *Scaling and renormalization in statistical physics*, Cambridge Lecture Notes in Physics 5, Cambridge University Press, Cambridge, 1996.
- [9] B. Duplantier and S. Sheffield: *Duality and the Knizhnik-Polyakov-Zamolodchikov relation in Liouville quantum gravity*, Phys. Rev. Lett. 102 (2009), 150603, 4 pp.
- [10] B. Duplantier and S. Sheffield: *Liouville Quantum Gravity and KPZ*, ArXiv [math.PR] 0808.1560. (2008)
- [11] D. Carpentier, P. Le Doussal: *Glass transition of a particle in a random potential, front selection in non linear RG and entropic phenomena in Liouville and SinhGordon models*, Phys. Rev. E 63, 026110 (2001)
- [12] Y.V. Fyodorov, J.P. Bouchaud: *Freezing and extreme value statistics in a Random Energy Model with logarithmically correlated potential* J. Phys. A: Math. Theor. 41 372001 (2008)
- [13] P. Le Doussal, Y.V. Fyodorov, A. Rosso, *Statistical Mechanics of Logarithmic REM: Duality, Freezing and Extreme Value Statistics of  $1/f$  Noises generated by Gaussian Free Fields*, arXiv:0907.2359 (2009)
- [14] D. Jerison and C. Kenig: *Hardy spaces,  $A_\infty$ , and singular integrals on chord-arc domains.*, Math. Scand. 50 (1982), 221–247.

- [15] P. Jones and S. Smirnov: *Removability theorems for Sobolev functions and quasiconformal maps*, Arkiv för Matematik 38 (2000), 263–279.
- [16] J.-P. Kahane: *Sur le chaos multiplicatif*, Ann. Sci. Math. Québec 9 (1985), 435–444.
- [17] J.-P. Kahane: *Some random series of functions*. Second edition. Cambridge Studies in Advanced Mathematics, 5. Cambridge University Press, 1985.
- [18] J.-P. Kahane: *Positive martingales and random measures*, Chi. Annal. Math 8B (1987), 1–12.
- [19] J.-P. Kahane and J. Peyrière: *Sur certaines martingales de Benoit Mandelbrot*, Advances in Math. 22 (1976), 131–145.
- [20] M. Lehtinen: *The dilatation of Beurling-Ahlfors extensions of quasisymmetric functions*, Ann. Acad. Sci. Fenn. 8 (1983), 187–191.
- [21] O. Lehto: *Homeomorphisms with a given dilatation*, in: Proceedings of the Fifteenth Scandinavian Congress (Oslo, 1968), pp. 58–73. Lecture Notes in Mathematics 118, Springer, 1970.
- [22] B. Mandelbrot: *Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier*. Journal of Fluid Mechanics 62 (1974), 331–358
- [23] G. M. Molchan: *Scaling exponents and multifractal dimensions for independent random cascades*, Comm. Math. Phys. 179 (1996), 681–702.
- [24] K. Oikawa: *Welding of polygons and the type of Riemann surfaces*, Kodai Math. Sem. Rep. 13 (1961), 37–52.
- [25] T. Reed: *On the boundary correspondence of quasiconformal mappings of domains bounded by quasicircles*. Pacific J. Math. 28 (1969), 653–661.
- [26] R. Robert and V. Vargas: *Gaussian multiplicative chaos revisited*, ArXiv [math.PR] 0807.1030. (2008).
- [27] G. Samorodnitsky: *Probability tails of Gaussian extrema*, Stochastic Process. Appl. 38 (1991), 55–84.
- [28] O. Schramm: *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel J. Math. 118 (2000), 221–288.
- [29] O. Schramm: *Conformally invariant scaling limits: an overview and a collection of problems*. In: International Congress of Mathematicians. Vol. I, 513–543, Eur. Math. Soc., 2007.
- [30] S. Smirnov: *Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model*, to appear in Annals of Math.
- [31] M. Talagrand: *Sharper bounds for Gaussian and empirical processes*, Ann. Prob. 22 (1994), 28–76.
- [32] J. Vainio: *Conditions for the possibility of conformal sewing*. Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 53 (1985), 43 pp.
- [33] M. Vuorinen: *Conformal geometry and quasiregular mappings*. Lecture Notes in Mathematics 1319. Springer, 1988.

UNIVERSITY OF HELSINKI, DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. Box 68 ,  
 FIN-00014 UNIVERSITY OF HELSINKI, FINLAND  
*E-mail address:* kari.astala@helsinki.fi

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, 10 HILLHOUSE AVE, NEW HAVEN, CT,  
 06510, U.S.A.  
*E-mail address:* jones@math.yale.edu

UNIVERSITY OF HELSINKI, DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. Box 68 ,  
 FIN-00014 UNIVERSITY OF HELSINKI, FINLAND  
*E-mail address:* antti.kupiainen@helsinki.fi

UNIVERSITY OF HELSINKI, DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. Box 68 ,  
 FIN-00014 UNIVERSITY OF HELSINKI, FINLAND  
*E-mail address:* eero.saksman@helsinki.fi